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Analysis of variance of balanced fractional factorial designs

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Abstract

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This paper surveys the analysis of variance (ANOVA) of balanced fractional factorial designs derived from balanced arrays. Attention is focused on ANOVA obtained by use of the algebraic structure of the triangular multidimensional partially balanced association scheme and the multidimensional relationship.

1. Introduction

The analysis of variance (ANOVA) is a statistical technique for analyzing measurements, which depend on several kinds of effects operating simultaneously, to decide which kinds of effects are important and to estimate these effects. The term 'ANOVA' in statistics was introduced by Fisher [5–7]. The ANOVA of balanced fractional 2^m factorial (2^m-BFF) designs of odd and even resolution and 3^m-BFF designs of resolution V has been studied by Kuwada [16–19]. The ANOVA and the testing hypotheses of various designs have been considered by many researchers (e.g., Scheffé [22], and Seber [24]).

As a generalization of an orthogonal array, the concept of a balanced array (B-array) was first introduced by Chakravarti [3] as a partially balanced array. It was later called a 'balanced array' in Srivastava and Chopra [31] since it is a generalization of a BIB design rather than a PBIB design. Under some conditions, a B-array of strength t, size N, with m constraints, s levels and index set $\{\mu_{i_0 i_1 \cdots i_{s-1}}\}$, written BA$(N, m, s, t; \{\mu_{i_0 i_1 \cdots i_{s-1}}\})$ for brevity, gives an s^m-BFF design (e.g., Kuwada and/or Nishii [13, 20], Srivastava [29], and Yamamoto, Shirakura and Kuwada [34]).

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A BFF design has been studied by Hoke [9, 10], Kuwada and/or Nishii [12, 14, 15, 21], Shirakura and/or Kuwada [25–28], Srivastava and Ariyaratna [30], Srivastava and/or Chopra [29, 31–33], Yamamoto, Shirakura and Kuwada [35], and so on.

This paper surveys the ANOVA of 2^m-BFF designs of odd and even resolution and 3^m-BFF designs of resolution V and IV. The focus is on ANOVA obtained by using the algebraic structure of a triangular multidimensional partially balanced (TMDPB) association scheme and a multidimensional relationship (MDR). The concept of the MDPB association scheme was introduced by Bose and Srivastava [2] as a generalization of an ordinary association scheme, and the MDR is regarded as a generalization of a relationship which was first introduced by James [11] in an experimental design. In Section 3, we consider the cases of 2^m-BFF designs of resolution $2l+1$ and $2l$. Section 4 gives the ANOVA of 3^m-BFF designs of resolution V and IV using the MDR and its algebra. In Sections 3 and 4, we consider the ANOVA from the algebraic point of view, i.e., the decomposition of an N-dimensional vector space. Section 5 presents three unsolved problems regarding the ANOVA of a BFF design.

2. Preliminaries

The ANOVA is a standard technique for handling the data (or observations) in an experiment. In various cases of the ANOVA, the total sum of squares (SS) is partitioned into a sum of other SS's. This corresponds to a decomposition of the identity matrix in matrix theory. The following are well-known results ([8]).

Lemma 2.1. *Let A_i $(1 \leqslant i \leqslant k)$ be real and symmetric matrices of order n such that $I_n = \sum_{i=1}^k A_i$ and $\operatorname{rank}(A_i) = r_i$, where I_p is the identity matrix of order p. Then the following are equivalent:*

(1) $\sum_{i=1}^k r_i = n$,

(2) $A_i^2 = A_i$ *for* $1 \leqslant i \leqslant k$,

(3) $A_i A_j = 0_{n \times n}$ *for* $i \neq j$,

where $0_{p \times q}$ is the $p \times q$ matrix of zeros.

Lemma 2.1 yields the following immediately.

Theorem 2.1 (Cochran's Theorem). *Let y be an $n \times 1$ random vector distributed as $\mathcal{N}(\mathbf{0}_n, \sigma^2 I_n)$, where $\mathbf{0}_p = 0_{p \times 1}$, and suppose A_i $(1 \leqslant i \leqslant k)$ are real and symmetric matrices of order n such that $\sum_{i=1}^k A_i = I_n$ and $\operatorname{rank}(A_i) = r_i$. Then the quadratic forms $y' A_i y$ have independent χ^2 distributions with r_i degrees of freedom if and only if $\sum_{i=1}^k r_i = n$, where A' denotes the transpose of the matrix A.*

Note that in matrix notation, Cochran's theorem may be written as follows: Let A_i $(1 \leqslant i \leqslant k)$ be matrices as in Lemma 2.1. Then a necessary and sufficient condition for

$A_iA_j = \delta_{ij}A_i$, i.e., A_i being orthogonal projections, is that $\sum_{i=1}^{k} r_i = n$, where δ_{ab} denotes Kronecker's delta.

3. ANOVA of 2^m-BFF designs

3.1. 2^m-BFF designs of resolution $2l+1$

Consider a fractional 2^m factorial (2^m-FF) experiment, where $m \geqslant 4$. Let T be a fraction with N assemblies (or treatment combinations). Then T can be expressed as a (0, 1)-matrix of size $N \times m$, whose rows denote the assemblies. We shall consider the situation in which $(l+1)$-factor and higher order interactions are assumed to be negligible, where $2l \leqslant m$. Then the linear model is given by

$$y(T) = E_T\Theta + e_T, \tag{3.1}$$

where $y(T)$ is a vector of N observations, E_T is the design matrix of size $N \times v_l$, $\Theta' = (\{\theta_0\}; \{\theta_t\}; \ldots; \{\theta_{t_1 \cdots t_l}\})$, and e_T is an error vector distributed as $\mathcal{N}(\mathbf{0}_N, \sigma^2 I_N)$. Here $v_l = 1 + \binom{m}{1} + \cdots + \binom{m}{l}$ and $N \geqslant v_l$. The normal equation for estimating Θ is given by $M_T\hat{\Theta} = E_T'y(T)$, where $M_T = E_T'E_T$ is called the information matrix of order v_l. If M_T is nonsingular, the BLUE of Θ and its variance-covariance matrix are given by $\hat{\Theta} = M_T^{-1}E_T'y(T)$ and $\mathrm{Var}[\hat{\Theta}] = \sigma^2 M_T^{-1}$, respectively.

A relation between a 2^m-BFF design and a B-array was obtained by Yamamoto, Shirakura and Kuwada [34] as follows.

Proposition 3.1. *A necessary and sufficient condition for a 2^m-FF design T of resolution $2l+1$ to be balanced is that T is a* $\mathrm{BA}(N, m, 2, 2l; \{\mu_{i_1}\})$, *where $\mu_{i_1} = \mu_{i_0 i_1}$, provided the information matrix M_T is nonsingular.*

Using the properties of the TMDPB association scheme and its algebra, the information matrix M_T of T, being a $\mathrm{BA}(N, m, 2, 2l; \{\mu_i\})$. is isomorphic to $\|\kappa_\beta^{u,v}\|$ $(= K_\beta$, say) of order $(l-\beta+1)$ with multiplicities $\binom{m}{\beta} - \binom{m}{\beta-1}$ $(=\phi_\beta$, say), where

$$\kappa_\beta^{u,v} = \kappa_\beta^{v,u} = \sum_{\alpha=0}^{\beta+u} z_{\beta\alpha}^{(\beta+u,\beta+v)}\gamma_{v-u+2\alpha} \quad \text{for } 0 \leqslant u \leqslant v \leqslant l-\beta \text{ and } 0 \leqslant \beta \leqslant l, \tag{3.2}$$

$$z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} (-1)^{\alpha-b}\binom{u-\beta}{b}\binom{u-b}{u-\alpha}\binom{m-u-\beta+b}{b} \times \left\{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}\right\}^{1/2} \Big/ \binom{v-u+b}{b} \quad \text{for } u \leqslant v,$$

$$\gamma_i = \sum_{j=0}^{2l}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{2l-i}{j-i+p}\mu_j \quad \text{for } 0 \leqslant i \leqslant 2l \tag{3.3}$$

(see [34, 35]). Thus we can easily prove the following (see [35]).

Proposition 3.2. *Let T be a* $\mathrm{BA}(N, m, 2, 2l; \{\mu_i\})$. *The information matrix M_T is nonsingular, i.e., T is of resolution $2l+1$, if and only if every K_β is positive definite for $0 \leqslant \beta \leqslant l$.*

Proposition 3.2 will be used to obtain the bases of some algebra in the next section.

3.2. *Structure of TMDPB association algebras*

Now we consider a 2^m-BFF design of resolution $2l+1$ derived from a $\mathrm{BA}(N, m, 2, 2l; \{\mu_i\})$, where $N > v_l$. Let

$$F_\beta^{u,v} = E_u A_\beta^{(u,v)\#} E_v' \quad \text{for } \beta \leqslant u, v \leqslant l \text{ and } 0 \leqslant \beta \leqslant l, \tag{3.4}$$

where E_k $(0 \leqslant k \leqslant l)$ are $N \times \binom{m}{k}$ submatrices of E_T corresponding to $(\{\theta_{t_1 \cdots t_k}\})$ $(=\Theta_k'$, say), i.e., $E_T = [E_0; E_1; \ldots; E_l]$,

$$A_\beta^{(u,v)\#} = \{A_\beta^{(v,u)\#}\}' = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} A_\alpha^{(u,v)} \quad \text{for } \beta \leqslant u \leqslant v \leqslant l, \tag{3.5}$$

$$A_\alpha^{(u,v)} = \{A_\alpha^{(v,u)}\}' = \sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)} A_\beta^{(u,v)\#} \quad \text{for } \alpha \leqslant u \leqslant v \leqslant l,$$

$$z_{(u,v)}^{\beta\alpha} = \phi_\beta z_{\beta\alpha}^{(u,v)} \Bigg/ \left\{ \binom{m}{u} \binom{u}{\alpha} \binom{m-u}{v-u+\alpha} \right\} \quad \text{for } u \leqslant v$$

(see [28, 35]). Here $A_\alpha^{(u,v)}$'s are $\binom{m}{u} \times \binom{m}{v}$ local association matrices of the TMDPB association scheme (e.g., [34, 35]). Some properties of $A_\beta^{(u,v)\#}$ $(\beta \leqslant u,\ v \leqslant l,\ 0 \leqslant \beta \leqslant l)$ are

$$A_0^{(u,v)\#} = \left[1 \Bigg/ \left\{ \binom{m}{u} \binom{m}{v} \right\}^{1/2} \right] G_{\binom{m}{u} \times \binom{m}{v}}, \tag{3.6}$$

$$\sum_{\beta=0}^{u} A_\beta^{(u,u)\#} = I_{\binom{m}{u}}, \tag{3.7}$$

$$A_\beta^{(u,w)\#} A_\gamma^{(w,v)\#} = \delta_{\beta\gamma} A_\beta^{(u,v)\#}, \tag{3.8}$$

$$\operatorname{rank}(A_\beta^{(u,v)\#}) = \phi_\beta, \tag{3.9}$$

where $G_{p \times q}$ is the $p \times q$ matrix with all unity. It holds that

$$E_i' E_j = \sum_{\beta=0}^{\min(i,j)} \kappa_\beta^{i-\beta, j-\beta} A_\beta^{(i,j)\#} \tag{3.10}$$

(e.g., [35]). Hence from (3.4) and (3.8), we get

$$F_\beta^{u,w} F_\gamma^{s,v} = \delta_{\beta\gamma} \kappa_\beta^{w-\beta, s-\beta} F_\beta^{u,v}, \tag{3.11}$$

where $\kappa_\beta^{u,v}$'s are given by (3.2). Thus by (3.11), the left regular representations of the algebra $[F_\beta^{u,v} \mid \beta \leqslant u, v \leqslant l,\ 0 \leqslant \beta \leqslant l]$ ($=\mathscr{A}$, say) are given by

$$F_\beta^{u,v} F'_\beta = F'_\beta \{ I_{l-\beta+1} \otimes H_\beta^{u,v} \} \quad \text{for } \beta \leqslant u, v \leqslant l \text{ and } 0 \leqslant \beta \leqslant l,$$

where $F'_\beta = (F_\beta^{\beta,\beta}, F_\beta^{\beta+1,\beta}, \ldots, F_\beta^{l,\beta}, \ldots, F_\beta^{l,l})$, $H_\beta^{u,v}$'s are of order $(l-\beta+1)$, and $A \otimes B$ denotes the Kronecker product. Here the (a,b)-elements, $h_\beta^{u,v}(a,b)$, say, of $H_\beta^{u,v}$ are given by

$$h_\beta^{u,v}(a,b) = \begin{cases} \kappa_\beta^{u-\beta,b} & \text{if } a = u-\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Letting $\mathscr{A}_\beta = [F_\beta^{u,v} \mid \beta \leqslant u, v \leqslant l]$ for $0 \leqslant \beta \leqslant l$, Kuwada [19] has proved the following.

Theorem 3.1. (i) *The matrix algebras $\mathscr{A}_\beta$ generated by $F_\beta^{u,v}$ ($\beta \leqslant u, v, \leqslant l$) for $0 \leqslant \beta \leqslant l$ are minimal two-sided ideals of the algebra $\mathscr{A}$, and $\mathscr{A}_\beta \mathscr{A}_\gamma = \delta_{\beta\gamma} \mathscr{A}_\beta$.*

(ii) *The matrix algebra $\mathscr{A}$ generated by $F_\beta^{u,v}$ ($\beta \leqslant u, v \leqslant l$, $0 \leqslant \beta \leqslant l$) is decomposed into the direct sum of the ideals $\mathscr{A}_\beta$ of $\mathscr{A}$, i.e.,*

$$\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_l.$$

(iii) *Each ideal $\mathscr{A}_\beta$ has $f_\beta^{u-\beta, v-\beta}$ ($\beta \leqslant u, v \leqslant l$) as its bases, and it is isomorphic to the complete $(l-\beta+1) \times (l-\beta+1)$ matrix algebra with multiplicity ϕ_β for $0 \leqslant \beta \leqslant l$, where*

$$f_\beta^{u,v} = \sum_{p=0}^{l-\beta} \kappa_{v,p}^\beta F_\beta^{\beta+u, \beta+p} \quad \textit{for } 0 \leqslant u, v \leqslant l-\beta \textit{ and } 0 \leqslant \beta \leqslant l, \tag{3.13}$$

$$K_\beta^{-1} = \| \kappa_{a,b}^\beta \|.$$

Sketch of proof. The formula (3.11) shows that (i) and (ii) hold. From (3.12), we have $F_\beta^u = K_\beta f_\beta^u$, where $F_\beta^u = (F_\beta^{u,\beta}, F_\beta^{u,\beta+1}, \ldots, F_\beta^{u,l})'$ and $f_\beta^u = (f_\beta^{u-\beta,0}, f_\beta^{u-\beta,1}, \ldots, f_\beta^{u-\beta,l-\beta})'$. Thus from Proposition 3.2, we get $f_\beta^u = K_\beta^{-1} F_\beta^u$, which yields (iii). This completes the proof. □

Let

$$\sum_{k=0}^{r} P_\beta^{\beta+k} = \sum_{i=0}^{r} \left[f_\beta^{i,i} + \sum_{q=r+1}^{l-\beta} \left\{ \sum_{j=0}^{r} \eta_\beta^{i,j}(r) \kappa_\beta^{q,j} \right\} f_\beta^{i,q} \right]$$
$$\text{for } 0 \leqslant r \leqslant l-\beta \text{ and } 0 \leqslant \beta \leqslant l, \tag{3.14}$$

where

$$K_\beta(r)^{-1} = \begin{bmatrix} \eta_\beta^{0,0}(r) & \eta_\beta^{0,1}(r) & \cdots & \eta_\beta^{0,r}(r) \\ \eta_\beta^{1,0}(r) & \eta_\beta^{1,1}(r) & \cdots & \eta_\beta^{1,r}(r) \\ \vdots & \vdots & \cdots & \vdots \\ \eta_\beta^{r,0}(r) & \eta_\beta^{r,1}(r) & \cdots & \eta_\beta^{r,r}(r) \end{bmatrix},$$

$$K_\beta(r)=\begin{bmatrix} \kappa_\beta^{0,0} & \kappa_\beta^{0,1} & \cdots & \kappa_\beta^{0,r} \\ \kappa_\beta^{1,0} & \kappa_\beta^{1,1} & \cdots & \kappa_\beta^{1,r} \\ \vdots & \vdots & \cdots & \vdots \\ \kappa_\beta^{r,0} & \kappa_\beta^{r,1} & \cdots & \kappa_\beta^{r,r} \end{bmatrix}.$$

Then from (3.13) and (3.14), we get

$$P_\beta^{\beta+r}=\sum_{i=0}^{r}\sum_{j=0}^{r}\eta_\beta^{i,j}(r)F_\beta^{\beta+i,\beta+j}-\sum_{u=0}^{r-1}\sum_{v=0}^{r-1}\eta_\beta^{u,v}(r-1)F_\beta^{\beta+u,\beta+v}$$

$$\text{for } 0\leqslant r\leqslant l-\beta \text{ and } 0\leqslant\beta\leqslant l, \tag{3.15}$$

where $\eta_\beta^{u,v}(-1)=0$. Thus we have the following (see [19]).

Theorem 3.2. (i) *The* $P_\beta^{\beta+r}$ $(0\leqslant r\leqslant l-\beta,\ 0\leqslant\beta\leqslant l)$ *and* P_e *are mutually orthogonal idempotent matrices, i.e., mutually orthogonal projections, where*

$$P_e=I_N-\sum_{\beta=0}^{l}\sum_{r=0}^{l-\beta}P_\beta^{\beta+r}.$$

(ii) rank $(P_\beta^{\beta+r})=\phi_\beta$ *for* $0\leqslant r\leqslant l-\beta$ *and* $0\leqslant\beta\leqslant l$,

rank$(P_e)=N-\nu_l$.

Sketch of proof. From (3.8), (3.11) and (3.15), $P_\beta^{\beta+r}$'s are mutually orthogonal idempotent, and hence (i) holds. While (3.9) and (i) imply (ii). Thus the proof is completed. □

3.3. ANOVA of 2^m-BFF designs of resolution $2l+1$

Using the properties of $A_\beta^{(u,u)\#}$ as in (3.7), we have

$$\Theta_k=\sum_{\beta=0}^{k}A_\beta^{(k,k)\#}\Theta_k \quad \text{for } 0\leqslant k\leqslant l.$$

Hence the linear model (3.1) is rewritten as

$$y(T)=\sum_{\beta=0}^{l}\sum_{k=\beta}^{l}E_kA_\beta^{(k,k)\#}\Theta_k+e_T.$$

From (3.6), (3.8) and (3.9), we note that (i) every element of the vector $A_0^{(k,k)\#}\Theta_k$ $(0\leqslant k\leqslant l)$ represents the average of the effects of the k-factor interactions, (ii) the elements of $A_\beta^{(k,k)\#}\Theta_k$ $(1\leqslant\beta\leqslant k)$ represent the contrasts among these effects, (iii) any two contrasts, one belonging to $A_\beta^{(k,k)\#}\Theta_k$ and the other to $A_\gamma^{(k,k)\#}\Theta_k$ $(2\leqslant\beta\neq\gamma\leqslant k)$, are orthogonal, and (iv) there are ϕ_β independent parametric functions of Θ_k in $A_\beta^{(k,k)\#}\Theta_k$ $(0\leqslant\beta\leqslant k)$.

It is empirically known that the lower order interactions (including the main effects) are more important than the higher order interactions. Thus we are first interested in testing the hypotheses H_β^l against K_β^l $(0\leqslant\beta\leqslant l)$, where H_β^l's and K_β^l's are the hypotheses that $A_\beta^{(l,l)\#}\Theta_l=\mathbf{0}_{\binom{m}{l}}$ and $A_\beta^{(l,l)\#}\Theta_l\neq\mathbf{0}_{\binom{m}{l}}$, respectively. Note that $\bigcap_{\gamma=0}^{l} H_\gamma^l$ and $\bigcap_{\gamma=1}^{l} H_\gamma^l$ mean that $\Theta_l=\mathbf{0}_{\binom{m}{l}}$ and $\Theta_l=c_l\mathbf{1}_{\binom{m}{l}}$, respectively, where c_l is a constant and $\mathbf{1}_p=G_{p\times 1}$. Next if H_β^l is accepted for some β $(0\leqslant\beta\leqslant l-1)$, then we wish to test the hypothesis $H_\beta^{l-1\,l}$ against H_β^l, where $H_\beta^{l-1\,l}$ is the hypothesis that $A_\beta^{(k,k)\#}\Theta_k=\mathbf{0}_{\binom{m}{k}}$ for $l-1\leqslant k\leqslant l$. If $H_\beta^{l-1\,l}$ is accepted for some β $(0\leqslant\beta\leqslant l-2)$, we consider testing of the hypothesis $H_\beta^{l-2\,l-1\,l}$ against $H_\beta^{l-1\,l}$, where $H_\beta^{l-2\,l-1\,l}$ is hypothesis that $A_\beta^{(k,k)\#}\Theta_k=\mathbf{0}_{\binom{m}{k}}$ for $l-2\leqslant k\leqslant l$, and so on. This is the nested test procedure in which we accept $H_\beta^{i\cdots l}$ against K_β^l for fixed β $(0\leqslant\beta\leqslant i)$ only if all the tests H_β^l against K_β^l, $H_\beta^{l-1\,l}$ against $H_\beta^l,\ldots,$ and $H_\beta^{i\cdots l}$ against $H_\beta^{i+1\cdots l}$ are not significant, where $H_\beta^{i\cdots l}=\bigcap_{k=j}^{l} H_\beta^k$ $(0\leqslant\beta\leqslant j)$ and H_β^k's are the hypotheses that $A_\beta^{(k,k)\#}\Theta_k=\mathbf{0}_{\binom{m}{k}}$ for $j\leqslant k\leqslant l$. The nested method is best for a certain class of reasonable test procedures (e.g., Anderson [1], and Seber [23]).

It follows from Theorem 3.2 that

$$\mathscr{R}^N=\mathscr{R}_0^0\oplus\mathscr{R}_0^1\oplus\cdots\oplus\mathscr{R}_0^l\oplus\mathscr{R}_1^1\oplus\mathscr{R}_1^2\oplus\cdots\oplus\mathscr{R}_1^l\oplus\cdots\oplus\mathscr{R}_l^l\oplus\mathscr{R}_e. \tag{3.16}$$

where

$$\mathscr{R}_\beta^{\beta+r}=\mathscr{R}_{R(r-1;\beta)^\perp}(R(r;\beta))\quad\text{for } 0\leqslant r\leqslant l-\beta \text{ and } 0\leqslant\beta\leqslant l,$$

$$R(r;\beta)=[E_\beta A_\beta^{(\beta,\beta)\#};\ E_{\beta+1}A_\beta^{(\beta+1,\beta+1)\#};\ldots;E_{\beta+r}A_\beta^{(\beta+r,\beta+r)\#}],$$

$$R(-1;\beta)=[\mathbf{0}_N],$$

$$\mathscr{R}_e=\mathscr{R}_{E_T^\perp}.$$

Here $\mathscr{R}^N$, $\mathscr{R}(A)$, $\mathscr{R}_{B^\perp}(C)$ and $\mathscr{R}_{D^\perp}$ denote an N-dimensional vector space, the linear subspace spanned by the column vectors of the matrix A, the orthocomplement subspace of $\mathscr{R}(B)$ relative to $\mathscr{R}(C)$ for the case $\mathscr{R}(B)\subset\mathscr{R}(C)$, and the orthocomplement subspace of $\mathscr{R}(D)$ relative to $\mathscr{R}^N$, respectively. Note that $P_\beta^{\beta+r}$ $(0\leqslant r\leqslant l-\beta,\ 0\leqslant\beta\leqslant l)$ and P_e are projections of $\mathscr{R}^N$ onto $\mathscr{R}_\beta^{\beta+r}$ and $\mathscr{R}_e$, respectively. Thus letting

$$S_\beta^{\beta+r}=\boldsymbol{y}(T)'P_\beta^{\beta+r}\boldsymbol{y}(T)\quad\text{for } 0\leqslant r\leqslant l-\beta \text{ and } 0\leqslant\beta\leqslant l,$$

$$S_e=\boldsymbol{y}(T)'P_e\boldsymbol{y}(T),$$

the following theorems can be proved easily (see [19]).

Theorem 3.3. $\boldsymbol{y}(T)'\boldsymbol{y}(T)=\sum_{\beta=0}^{l}\sum_{r=0}^{l-\beta}S_\beta^{\beta+r}+S_e$.

Theorem 3.4. *An unbiased estimator of* σ^2 *is given by* $\hat{\sigma}^2=S_e/(N-\nu_l)$.

Lemma 3.1. *Let*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} (=A, \textit{ say})$$

be a symmetric and positive definite matrix of order n ($\geqslant 2$), *where* A_{11}, $A_{12}=A'_{21}$ *and* A_{22} *are of size* $(n-1)\times(n-1)$, $(n-1)\times 1$ *and* 1×1, *respectively. Further let*

$$A^{-1}=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

$\boldsymbol{x}'=(\boldsymbol{x}'_0, x)$ *and* $\boldsymbol{y}'=(\boldsymbol{y}'_0, y)$, *where* B_{11}, $B_{12}=B'_{21}$ *and* B_{22} *are of size* $(n-1)\times(n-1)$, $(n-1)\times 1$ *and* 1×1, *respectively, and* $\boldsymbol{x}_0$ *and* $\boldsymbol{y}_0$ *are* $(n-1)\times 1$ *vectors. Then*

$$\boldsymbol{x}'A^{-1}\boldsymbol{y}-\boldsymbol{x}'_0A_{11}^{-1}\boldsymbol{y}_0$$
$$=\left\{\det\begin{bmatrix} A_{11} & A_{12} \\ \boldsymbol{x}'_0 & x \end{bmatrix}\right\}\left\{\det\begin{bmatrix} A_{11} & \boldsymbol{y}_0 \\ A_{21} & y \end{bmatrix}\right\}\Big/[\{\det(A)\}\{\det(A_{11})\}],$$

where $\det(A)$ *denotes the determinant of the matrix* A.

Proof. After some calculations, we get

$$B_{11}=A_{11}^{-1}+A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1}, \qquad B_{12}=-A_{11}^{-1}A_{12}B_{22}=B'_{21},$$
$$B_{22}=(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}.$$

Thus

$$\boldsymbol{x}'A^{-1}\boldsymbol{y}-\boldsymbol{x}'_0A_{11}^{-1}\boldsymbol{y}_0$$
$$=\boldsymbol{x}'_0B_{11}\boldsymbol{y}_0+xB_{21}\boldsymbol{y}_0+y\boldsymbol{x}'_0B_{12}+xyB_{22}-\boldsymbol{x}'_0A_{11}^{-1}\boldsymbol{y}_0$$
$$=\boldsymbol{x}'_0(A_{11}^{-1}+A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1})\boldsymbol{y}_0-xB_{22}A_{21}A_{11}^{-1}\boldsymbol{y}_0-y\boldsymbol{x}'_0A_{11}^{-1}A_{12}B_{22}$$
$$+xyB_{22}-\boldsymbol{x}'_0A_{11}^{-1}\boldsymbol{y}_0$$
$$=\{xy-xA_{21}A_{11}^{-1}\boldsymbol{y}_0-y\boldsymbol{x}'_0A_{11}^{-1}A_{12}+(\boldsymbol{x}'_0A_{11}^{-1}A_{12})(A_{21}A_{11}^{-1}\boldsymbol{y}_0)\}B_{22}$$
$$=(x-\boldsymbol{x}'_0A_{11}^{-1}A_{12})(y-A_{21}A_{11}^{-1}\boldsymbol{y}_0)/(A_{22}-A_{21}A_{11}^{-1}A_{12})$$
$$=\left\{\det\begin{bmatrix} A_{11} & A_{12} \\ \boldsymbol{x}'_0 & x \end{bmatrix}\right\}\left\{\begin{bmatrix} A_{11} & \boldsymbol{y}_0 \\ A_{21} & y \end{bmatrix}\right\}\Big/[\{\det(A)\}\{\det(A_{11})\}],$$

which is the required result. □

The noncentrality parameters, $\lambda_\beta^{\beta+r}/\sigma^2$, say, of the quadratic forms $\boldsymbol{y}(T)'P_\beta^{\beta+r}\boldsymbol{y}(T)/\sigma^2$ $(0\leqslant r\leqslant l-\beta,\ 0\leqslant\beta\leqslant l)$ are defined by

$$\lambda_\beta^{\beta+r}/\sigma^2=\mathrm{Exp}[\boldsymbol{y}(T)']P_\beta^{\beta+r}\mathrm{Exp}[\boldsymbol{y}(T)]/\sigma^2,$$

where $\mathrm{Exp}[\boldsymbol{y}]$ denotes the expected value of a random vector $\boldsymbol{y}$. We have the following theorem due to Kuwada [19].

Theorem 3.5. *The noncentrality parameters of the quadratic forms* $y(T)'P_\beta^{\beta+r}y(T)/\sigma^2$ $(0\leqslant r\leqslant l-\beta,\ 0\leqslant\beta\leqslant l)$ *are given by*

$$\lambda_\beta^{\beta+r}/\sigma^2=\sum_{a=\beta+r}^{l}\sum_{b=\beta+r}^{l}\{c_\beta(a,b;r)/\sigma^2\}\,\Theta_a' A_\beta^{(a,b)\#}\,\Theta_b, \tag{3.17}$$

where

$$c_\beta(a,b;r)=\begin{cases}(\kappa_\beta^{a-\beta,0})(\kappa_\beta^{0,b-\beta})/\kappa_\beta^{0,0} & \text{if } r=0,\\ \left\{\det\begin{bmatrix}K_\beta(r-1) & C_{12}\\ x_0(a)' & x(a)\end{bmatrix}\right\}\left\{\det\begin{bmatrix}K_\beta(r-1) & y_0(b)\\ C_{21} & y(b)\end{bmatrix}\right\}\Big/ & \\ \qquad[\{\det(K_\beta(r))\}\{\det(K_\beta(r-1))\}] & \text{if } r\geqslant 1,\end{cases}$$

$$C_{12}=(\kappa_\beta^{0,r},\kappa_\beta^{1,r},\ldots,\kappa_\beta^{r-1,r})',\qquad C_{21}=(\kappa_\beta^{r,0},\kappa_\beta^{r,1},\ldots,\kappa_\beta^{r,r-1}),$$

$$x_0(a)=(\kappa_\beta^{a-\beta,0},\kappa_\beta^{a-\beta,1},\ldots,\kappa_\beta^{a-\beta,r-1})',\qquad x(a)=\kappa_\beta^{a-\beta,r},$$

$$y_0(b)=(\kappa_\beta^{0,b-\beta},\kappa_\beta^{1,b-\beta},\ldots,\kappa_\beta^{r-1,b-\beta})'\qquad\text{and}\qquad y(b)=\kappa_\beta^{r,b-\beta}.$$

Proof. From (3.4), (3.8) and (3.10), we get

$$\begin{aligned}
&\sum_{i=0}^{r}\sum_{j=0}^{r}\eta_\beta^{i,j}(r)\,E_a'F_\beta^{\beta+i,\beta+j}E_b\\
&=\sum_i\sum_j\eta_\beta^{i,j}(r)E_a'E_{\beta+i}A_\beta^{(\beta+i,\beta+j)\#}\,E_{\beta+j}'E_b\\
&=\sum_i\sum_j\eta_\beta^{i,j}(r)\kappa_\beta^{a-\beta,i}\kappa_\beta^{j,b-\beta}A_\beta^{(a,b)\#}\\
&=\begin{cases}\sum_i\delta_{i\,b-\beta}\kappa_\beta^{a-\beta,i}A_\beta^{(a,b)\#} & \text{if } b-\beta\leqslant r,\\ \sum_j\delta_{j\,a-\beta}\kappa_\beta^{j,b-\beta}A_\beta^{(a,b)\#} & \text{if } a-\beta\leqslant r,\\ \sum_i\sum_j\eta_\beta^{i,j}(r)\kappa_\beta^{a-\beta,i}\kappa_\beta^{j,b-\beta}A_\beta^{(a,b)\#} & \text{if } a-\beta>r \text{ and } b-\beta>r,\end{cases}\\
&=\begin{cases}\kappa_\beta^{a-\beta,b-\beta}A_\beta^{(a,b)\#} & \text{if } a-\beta\leqslant r \text{ or } b-\beta\leqslant r,\\ \sum_i\sum_j\eta_\beta^{i,j}(r)\kappa_\beta^{a-\beta,i}\kappa_\beta^{j,b-\beta}A_\beta^{(a,b)\#} & \text{if } a-\beta>r \text{ and } b-\beta>r.\end{cases}
\end{aligned}$$

Thus (3.15) yields

$$\begin{aligned}\lambda_\beta^{\beta+r}/\sigma^2=\Bigg[\sum_{a=\beta+r}^{l}\sum_{b=\beta+r}^{l}\Bigg\{&\sum_{i=0}^{r}\sum_{j=0}^{r}\eta_\beta^{i,j}(r)\kappa_\beta^{a-\beta,i}\kappa_\beta^{j,b-\beta}\\ &-\sum_{u=0}^{r-1}\sum_{v=0}^{r-1}\eta_\beta^{u,v}(r-1)\kappa_\beta^{a-\beta,u}\kappa_\beta^{v,b-\beta}\Bigg\}\,\Theta_a'A_\beta^{(a,b)\#}\,\Theta_b\Bigg]\Big/\sigma^2.\end{aligned}$$

Therefore, from Lemma 3.1, we have (3.17), which completes the proof. □

It follows from (3.17) that $H_\beta^{i\cdots l}$ $(0\leqslant\beta\leqslant i\leqslant l)$ means $\lambda_\beta^i=0$, and vice versa. Here we give the ANOVA table of 2^m-BFF designs of resolution $2l+1$ in Table 1.

Now we consider testing of hypotheses $H_\beta^{i\cdots l}$ against K_β^l $(\beta\leqslant i\leqslant l)$ for fixed β $(0\leqslant\beta\leqslant i)$. Then the test statistics for the nested method

$$\frac{S_\beta^l/\phi_\beta}{S_e/(N-v_l)}\quad(=F_\beta^l,\ \text{say}),\qquad \frac{S_\beta^{l-1}/\phi_\beta}{(S_e+S_\beta^l)/(N-v_l+\phi_\beta)}\quad(=F_\beta^{l-1l},\ \text{say}),$$

$$\ldots,\ \text{and}\quad \frac{S_\beta^i/\phi_\beta}{(S_e+S_\beta^{i+1}+\cdots+S_\beta^l)/\{N-v_l+(l-i)\phi_\beta\}}\quad(=F_\beta^{i\cdots l},\ \text{say})$$

all have F distributions and the nesting procedure is continued until a significant test is obtained. Here $F_\beta^{j\cdots l}$ $(\beta\leqslant j\leqslant l)$ are central or noncentral F distributions with ϕ_β and $\{N-v_l+(l-j)\phi_\beta\}$ degrees of freedom (d.f.) and noncentrality parameters λ_β^j/σ^2 depending on which of $H_\beta^{l\cdots j}$'s are true.

3.4. Example

To illustrate the usefulness of the results of Section 3.3, we present an example. Let

$$T=\begin{bmatrix} 0 & 1\,0\,0\,0 & 1\,0\,0\,0 & 1\,1\,1\,0\,0\,0 & 1\,1\,1\,0 & 1\,1 \\ 0 & 0\,1\,0\,0 & 0\,1\,0\,0 & 1\,0\,0\,1\,1\,0 & 1\,1\,0\,1 & 1\,1 \\ 0 & 0\,0\,1\,0 & 0\,0\,1\,0 & 0\,1\,0\,1\,0\,1 & 1\,0\,1\,1 & 1\,1 \\ 0 & 0\,0\,0\,1 & 0\,0\,0\,1 & 0\,0\,1\,0\,1\,1 & 0\,1\,1\,1 & 1\,1 \end{bmatrix},$$

which is a BA(21,4,2,4; $\{\mu_0=1,\ \mu_1=2,\ \mu_2=1,\ \mu_3=1,\ \mu_4=2\}$) and is also an optimal 2^4-BFF design of resolution V (see [32]). Then $\Theta_0=(\theta_0)$, $\Theta_1=(\theta_1,\theta_2,\theta_3,\theta_4)'$, $\Theta_2=(\theta_{12},\theta_{13},\theta_{14},\theta_{23},\theta_{24},\theta_{34})'$ and $v_2=11$. We further have

$$E_0=[+\quad ++++\quad ++++\quad ++++++\quad ++++\quad ++]',$$

$$E_1=\begin{bmatrix} - & +--- & +--- & +++--- & +++- & ++ \\ - & -+-- & -+-- & +--++- & ++-+ & ++ \\ - & --+- & --+- & -+-+-+ & +-++ & ++ \\ - & ---+ & ---+ & --+-++ & -+++ & ++ \end{bmatrix},$$

$$E_2=\begin{bmatrix} + & --++ & --++ & +----+ & ++-- & ++ \\ + & -+-+ & -+-+ & -+--+- & +-+- & ++ \\ + & -++- & -++- & --++-- & -++- & ++ \\ + & +--+ & +--+ & --++-- & +--+ & ++ \\ + & +-+- & +-+- & -+--+- & -+-+ & ++ \\ + & ++-- & ++-- & +----+ & --++ & ++ \end{bmatrix},$$

where $+$ and $-$ stand for 1 and -1, respectively. From (3.3), we have $N=\gamma_0=21$, $\gamma_1=-1$, $\gamma_2=1$, $\gamma_3=3$ *and* $\gamma_4=-3$, and hence from (3.2), $\kappa_0^{0,0}=21$, $\kappa_0^{0,1}=-2$,

Table 1
ANOVA of 2^m-BFF designs of resolution $2l+1$

Source	SS	d.f.	Noncentrality parameters
$A_{\beta}^{(i,i)\#}\Theta_i$	S_{β}^{i}	ϕ_{β}	$\lambda_{\beta}^{i}/\sigma^2$
Error	S_e	$N-\nu_l$	
Total	$y(T)'y(T)$	N	

for $0\leqslant\beta\leqslant i\leqslant l$.

$\kappa_0^{0,2}=\sqrt{6}$, $\kappa_0^{1,1}=24$, $\kappa_0^{1,2}=2\sqrt{6}$, $\kappa_0^{2,2}=22$, $\kappa_1^{0,0}=20$, $\kappa_1^{0,1}=-4\sqrt{2}$, $\kappa_1^{1,1}=24$ and $\kappa_2^{0,0}=16$. Thus $\eta_0^{0,0}(2)=126/2576$, $\eta_0^{0,1}(2)=14/2576$, $\eta_0^{0,2}(2)=-7\sqrt{6}/2576$, $\eta_0^{1,1}(2)=114/2576$, $\eta_0^{1,2}(2)=-11\sqrt{6}/2576$, $\eta_0^{2,2}(2)=125/2576$, $\eta_0^{0,0}(1)=24/500$, $\eta_0^{0,1}(1)=2/500$, $\eta_0^{1,1}(1)=21/500$, $\eta_0^{0,0}(0)=1/21$, $\eta_1^{0,0}(1)=6/112$, $\eta_1^{0,1}(1)=\sqrt{2}/112$, $\eta_1^{1,1}(1)=5/112$, $\eta_1^{0,0}(0)=1/20$ and $\eta_2^{0,0}(0)=1/16$. On the other hand, from the properties of the TMDPB association scheme, we have

$$A_0^{(0,0)}=[1],\qquad A_0^{(0,1)}=\mathbf{1}_4',\qquad A_0^{(0,2)}=\mathbf{1}_6',$$

$$A_0^{(1,1)}=I_4,\qquad A_1^{(1,1)}=G_{4\times4}-I_4,$$

$$A_0^{(1,2)}=\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix},\qquad A_0^{(1,2)}=G_{4\times6}-A_0^{(1,2)},$$

$$A_0^{(2,2)}=I_6,\qquad A_1^{(2,2)}=\begin{bmatrix}0&1&1&1&1&0\\1&0&1&1&0&1\\1&1&0&0&1&1\\1&1&0&0&1&1\\1&0&1&1&0&1\\0&1&1&1&1&0\end{bmatrix},$$

$$A_2^{(2,2)}=G_{6\times6}-A_0^{(2,2)}-A_1^{(2,2)}\ .$$

Thus from (3.5), it follows that

$$A_0^{(0,0)\#}=A_0^{(0,0)},\qquad A_0^{(0,1)\#}=(1/2)A_0^{(0,1)},$$

$$A_0^{(0,2)\#}=(1/\sqrt{6})A_0^{(0,2)},\qquad A_0^{(1,1)\#}=(1/4)\{A_0^{(1,1)}+A_1^{(1,1)}\},$$

$$A_1^{(1,1)\#}=(1/4)\{3A_0^{(1,1)}-A_1^{(1,1)}\},$$

$$A_0^{(1,2)\#}=(\sqrt{6}/12)\{A_0^{(1,2)}+A_1^{(1,2)}\},$$

$$A_1^{(1,2)\#}=(\sqrt{2}/4)\{A_0^{(1,2)}-A_1^{(1,2)}\},$$
$$A_0^{(2,2)\#}=(1/6)\{A_0^{(2,2)}+A_1^{(2,2)}+A_2^{(2,2)}\},$$
$$A_1^{(2,2)\#}=(1/2)\{A_0^{(2,2)}-A_2^{(2,2)}\},$$
$$A_2^{(2,2)\#}=(1/6)\{2A_0^{(2,2)}-A_1^{(2,2)}+2A_2^{(2,2)}\}.$$

Let $y(T)=(48,61,52,46,54,59,51,47,53,63,57,60,53,54,44,56,62,59,54,57,58)'$, which are fictitious data. Then we have

$$\hat{\Theta}_0=(141624/2576)\approx 54.978,$$
$$\hat{\Theta}_1=(1/2576)(10996,3268,-4092,1796)'$$
$$\approx(4.269,1.269,-1.589,0.697)',$$
$$\hat{\Theta}_2=(1/2576)(-2440,-876,-968,1056,-324,-1336)'$$
$$\approx(-0.947,-0.340,-0.376,0.410,-0.126,-0.519)'.$$

From Theorem 3.3, we get

$$S_0^0=y(T)'P_0^0y(T)=1317904/21\approx 62757.333,$$
$$S_0^1=y(T)'P_0^1y(T)=290521/2625\approx 110.675,$$
$$S_0^2=y(T)'P_0^2y(T)=746642/60375\approx 12.367,$$
$$S_1^1=y(T)'P_1^1y(T)=7556/20\approx 377.8,$$
$$S_1^2=y(T)'P_1^2y(T)=332/35\approx 9.486,$$
$$S_2^2=y(T)'P_2^2y(T)=28/3\approx 9.333,$$
$$S_e=y(T)'P_ey(T)=8534/161\approx 53.006.$$

Thus Theorem 3.4 shows

$$\hat{\sigma}^2=S_e/(N-\nu_2)=8534/(161\times 10)\approx 5.301.$$

First we consider the hypotheses H_β^2: $A_\beta^{(2,2)\#}\Theta_2=\mathbf{0}_6$, against K_β^2: $A_\beta^{(2,2)\#}\Theta_2\neq\mathbf{0}_6$ for $0\leqslant\beta\leqslant 2$. Then

$$F_\beta^2=\frac{S_\beta^2/\phi_\beta}{S_e/(N-\nu_2)}$$
$$=\begin{cases}\dfrac{28/(3\times 2)}{8534/(161\times 10)}\approx 0.880<F_{10}^2(0.05)\approx 4.10 & \text{if } \beta=2,\\ \dfrac{332/(35\times 3)}{8534/(161\times 10)}\approx 0.597<F_{10}^3(0.05)\approx 3.71 & \text{if } \beta=1,\\ \dfrac{746642/60375}{8534/(161\times 10)}\approx 2.333<F_{10}^1(0.05)\approx 4.96 & \text{if } \beta=0,\end{cases}$$

where $\phi_\beta = \binom{4}{\beta} - \binom{4}{\beta-1}$ for $0 \leqslant \beta \leqslant 2$, and $F^{n_1}_{n_2}(0.05)$ denotes the 5% point of the F distribution with n_1 and n_2 d.f. Therefore, all F^2_β $(0 \leqslant \beta \leqslant 2)$ with ϕ_β and 10 d.f. are not significant at the 5% level, i.e., $A^{(2,2)\#}_\beta \Theta_2 = \mathbf{0}_6$ for $0 \leqslant \beta \leqslant 2$. Thus we consider testing the hypotheses $H^{12}_\beta : A^{(k,k)\#}_\beta \Theta_k = \mathbf{0}_{\binom{4}{k}}$ for $1 \leqslant k \leqslant 2$, against H^2_β for $0 \leqslant \beta \leqslant 1$. Then

$$F^{12}_\beta = \frac{S^1_\beta/\phi_\beta}{(S_e + S^2_\beta)/(N - \nu_2 + \phi_\beta)}$$

$$= \begin{cases} \dfrac{(7556/20)/3}{(8534/161 + 332/35)/(10+3)} \approx 26.198 > F^3_{13}(0.05) & \text{if } \beta = 1, \\ \dfrac{290521/2625}{(8534/161 + 746642/60375)/(10+1)} \approx 18.623 > F^1_{11}\ (0.05) & \text{if } \beta = 0, \end{cases}$$

where $F^3_{13}(0.05) = 3.41$ and F^1_{11} $(0.05) = 4.84$, and hence F^{12}_1 with 3 and 13 d.f. and F^{12}_0 with 1 and 11 d.f. are significant at 5% level. Therefore we conclude that $A^{(2,2)\#}_\beta \Theta_2 = \mathbf{0}_6$ for $0 \leqslant \beta \leqslant 2$, i.e., $\Theta_2 = \mathbf{0}_6$, and the main effects θ_t $(1 \leqslant t \leqslant 4)$ are not always equal.

3.5. ANOVA of some 2^m-BFF designs of resolution $2l$

In this section, we consider a 2^m-BFF design of resolution $2l$ derived from a B-array. When $N \geqslant \nu_l$, there may exist a 2^m-BFF design of resolution $2l+1$ derived from a BA $(N, m, 2, 2l; \{\mu_i\})$. However, when $N = \nu_l$, there is no d.f. due to error. Thus we shall consider the case in which $N \leqslant \nu_l$. Such a design can be constructed by a BA $(N, m, 2, 2l; \{\mu_i\})$ satisfying the following conditions.

$$\det(K_\beta) \neq 0 \quad \text{for } 0 \leqslant \beta \leqslant l-1 \qquad \text{and} \qquad \det(K_l) = 0, \tag{3.18}$$

where $\nu_l - \{\binom{m}{l} - \binom{m}{l-1}\} < N \leqslant \nu_l$ (see Shirakura [25, 27]). Note that $\det(K_l) = 0$ means $\mu_l = 0$ (see [25]).In this case, the linear model (3.1) is rewritten as

$$y^*(T) = \sum_{\beta=0}^{l-1} \sum_{k=\beta}^{l} E_k A^{(k,k)\#}_\beta \Theta_k + e^*_T,$$

where e^*_T is an error vector distributed as $\mathcal{N}(\mathbf{0}_N, \sigma^{*2} I_N)$. Note that for a design satisfying (3.18), each effect in $(\Theta'_0, \Theta'_0, \ldots, \Theta'_{l-1})'$ and some linear combinations in Θ_l such that $A^{(l,l)\#}_\beta \Theta_l$ $(0 \leqslant \beta \leqslant l-1)$ are estimable (see [25]).

Let

$$P^{*\beta+r}_\beta = P^{\beta+r}_\beta \quad \text{for } 0 \leqslant r \leqslant l-\beta \text{ and } 0 \leqslant \beta \leqslant l-1$$

and

$$P^*_e = P_e + P^l_l \quad \left(= I_N - \sum_{\beta=0}^{l-1} \sum_{r=0}^{l-\beta} P^{*\beta+r}_\beta \right).$$

Then

$$\operatorname{rank}(P^{*\beta+r}_\beta) = \phi_\beta \quad \text{for } 0 \leqslant r \leqslant l-\beta \text{ and } 0 \leqslant \beta \leqslant l-1$$

and

$$\operatorname{rank}(P_e^*) = N - v_l + \binom{m}{l} - \binom{m}{l-1}.$$

Theorem 3.6.

$$y^*(T)'y^*(T) = \sum_{\beta=0}^{l-1} \sum_{r=0}^{l-\beta} S_\beta^{*\beta+r} + S_e^*,$$

where

$$S_\beta^{*\beta+r} = y^*(T)' P_\beta^{*\beta+r} y^*(T) \quad \textit{for } 0 \leqslant r \leqslant l-\beta \textit{ and } 0 \leqslant \beta \leqslant l-1,$$

$$S_e^* = y^*(T)' P_e^* y^*(T).$$

Theorem 3.7. *An unbiased estimator of* σ^{*2} *is give by*

$$\hat{\sigma}^{*2} = S_e^* \Big/ \left\{ N - v_l + \binom{m}{l} - \binom{m}{l-1} \right\}.$$

Using an argument similar to the one used in Section 3.3, the noncentrality parameters of $y^*(T)'P_\beta^{*\beta+r}y^*(T)/\sigma^{*2}$ $(0 \leqslant r \leqslant l-\beta,\ 0 \leqslant \beta \leqslant l-1)$ are given by $\lambda_\beta^{*\beta+r}/\sigma^{*2} = \lambda_\beta^{\beta+r}/\sigma^{*2}$, where $\lambda_\beta^{\beta+r}$'s are as in (3.17). Table 2 summarizes the observations.

Let $H_\beta^{*i\cdots l} = H_\beta^{i\cdots l}$ and $K_\beta^{*l} = K_\beta^l$ for $0 \leqslant \beta \leqslant l-1$ and $\beta \leqslant i \leqslant l$. Then the test statistics for the nested method

$$\frac{S_\beta^{*l}/\phi_\beta}{S_e^*/(N - v_l + \phi_l)} \quad (= F_\beta^{*l}, \text{ say}),$$

$$\frac{S_\beta^{*l-1}/\phi_\beta}{(S_e^* + S_\beta^{*l})/(N - v_l + \phi_l + \phi_\beta)} \quad (= F_\beta^{*l-1l}, \text{ say}), \ldots, \quad \text{and}$$

$$\frac{S_\beta^{*i}/\phi_\beta}{(S_e^* + S_\beta^{*i+1} + \cdots + S_\beta^{*l})/\{N - v_l + \phi_l + (l-i)\phi_\beta\}} \quad (= F_\beta^{*i\cdots l}, \text{ say}),$$

Table 2
ANOVA of 2^m-BFF designs of resolution $2l$

Source	SS	d.f.	Noncentrality parameters
$A_\beta^{(i,i)\#}\Theta_i$	S_β^{*i}	ϕ_β	$\lambda_\beta^{*i}/\sigma^{*2}$
Error	S_e^*	$N - v_l + \binom{m}{l} - \binom{m}{l-1}$	
Total	$y^*(T)'y^*(T)$	N	

for $\beta \leqslant i \leqslant l$ and $0 \leqslant \beta \leqslant l-1$.

have F distributions. This procedure is continued until a significant test is obtained for fixed β $(0 \leqslant \beta \leqslant l-1,\ \beta \leqslant i \leqslant l)$. Note that $F_{\beta}^{*j\ldots l}$ $(\beta \leqslant j \leqslant l,\ 0 \leqslant \beta \leqslant l-1)$ are noncentral F distributions with ϕ_β and $\{N - v_l + \phi_l + (l-j)\phi_\beta\}$ d.f. and noncentrality parameters $\lambda_\beta^{*j}/\sigma^{*2}$ if $H_\beta^{*j\ldots l}$'s are not true.

4. ANOVA of 3^m-BFF designs

4.1. 3^m-BFF designs of resolution V

Let T be a 3^m-FF design of resolution V with N assemblies, where $N \geqslant 1 + 2m^2$ $(= v(m)$, say) and $m \geqslant 4$. Then the linear model based on T is given by

$$\tilde{y}(T) = X_T \Gamma + \tilde{e}_T, \tag{4.1}$$

where $\tilde{y}(T)$ is a vector of N observations, X_T is the design matrix of size $N \times v(m)$, $\Gamma' = (\Gamma'_{00};\ \Gamma'_{10};\ \Gamma'_{01};\ \Gamma'_{20};\ \Gamma'_{02};\ \Gamma'_{11})$, and $\tilde{e}_T$ is an error vector distributed as $\mathcal{N}(\mathbf{0}_N, \tilde{\sigma}^2 I_N)$. Here $\Gamma'_{00} = (\{\Gamma(\phi)\})$, $\Gamma'_{10} = (\{\Gamma(t^1)\})$, $\Gamma'_{01} = (\{\Gamma(t^2)\})$, $\Gamma'_{20} = (\{\Gamma(t_1^1 t_2^1)\})$ $\Gamma'_{02} = (\{\Gamma(t_1^2 t_2^2)\})$ and $\Gamma'_{11} = (\{\Gamma(t_3^1 t_4^2)\})$, where $1 \leqslant t \leqslant m$, $1 \leqslant t_1 < t_2 \leqslant m$ and $1 \leqslant t_3 \neq t_4 \leqslant m$. In all our evaluations, we code the three symbols of a factor as 0, 1 or 2 and employ the standard orthogonal contrasts used in the 3^m case; viz., $-1, 0, 1$ and $1, -2, 1$ for the linear and the quadratic contrasts, respectively. The normal equation for estimating Γ is given by $\tilde{M}_T \hat{\Gamma} = X'_T \tilde{y}(T)$, where $\tilde{M}_T = X'_T X_T$. If $\tilde{M}_T$ is nonsingular, then the BLUE of Γ and its variance-covariance matrix are, respectively, given by $\hat{\Gamma} = \tilde{M}_T^{-1} X'_T \tilde{y}(T)$ and $\mathrm{Var}[\hat{\Gamma}] = \tilde{\sigma}^2 \tilde{M}_T^{-1}$. Under some conditions, a $\mathrm{BA}(N, m, 3, 4; \{\mu_{i_0 i_1 i_2}\})$ gives a 3^m-BFF design of resolution V, and vice versa (e.g., [12]).

Using the MDR and its algebra, the information matrix $\tilde{M}_T$ of a $\mathrm{BA}(N, m, 3, 4; \{\mu_{i_0 i_1 i_2}\})$, T, is isomorphic to $\| \xi_\beta^{a_1 a_2, b_1 b_2} \|$ $(= H_\beta$, say) for $0 \leqslant \beta \leqslant 2$ and $\| \xi_{f_{uv}}^{u_1 u_2, v_1 v_2} \|$ $(= H_f$, say) with multiplicities φ_β and φ_f, respectively, where $\varphi_0 = 1$, $\varphi_1 = m(m-3)/2$, $\varphi_2 = \binom{m-1}{2}$ and $\varphi_f = m-1$, and H_0, H_1, H_2 and H_f are symmetric matrices of order 6, 3, 1 and 6, respectively. Here a relationship between $\xi_\beta^{a_1 a_2, b_1 b_2}$'s (or $\xi_{f_{uv}}^{u_1 u_2, v_1 v_2}$'s) and $\gamma_{p_0 p_1 p_2}$'s is given by

$$\xi_0^{00,00} = \gamma_{400} = N, \quad \xi_0^{00,10} = \sqrt{m}\gamma_{310}, \quad \xi_0^{00,01} = \sqrt{m}\gamma_{301},$$

$$\xi_0^{00,20} = \sqrt{\binom{m}{2}}\gamma_{220}, \quad \xi_0^{00,02} = \sqrt{\binom{m}{2}}\gamma_{202}, \quad \xi_0^{00,11} = \sqrt{2\binom{m}{2}}\gamma_{211},$$

$$\xi_0^{10,10} = \{2\gamma_{400} + \gamma_{301} + 3(m-1)\gamma_{220}\}/3, \quad \xi_0^{10,01} = \gamma_{310} + (m-1)\gamma_{211},$$

$$\xi_0^{10,20} = \sqrt{(m-1)/2}\,\{4\gamma_{310} + 2\gamma_{211} + 3(m-2)\gamma_{130}\}/3,$$

$$\xi_0^{10,02} = \sqrt{(m-1)/2}\{2\gamma_{211} + (m-2)\gamma_{211}\},$$

$$\xi_0^{10,11} = \sqrt{m-1}\{2\gamma_{301} + \gamma_{202} + 3\gamma_{220} + 3(m-2)\gamma_{121}\}/3,$$

$$\xi_0^{01,01}=2\gamma_{400}-\gamma_{301}+(m-1)\gamma_{202},$$

$$\xi_0^{01,20}=\sqrt{(m-1)/2}\{2\gamma_{220}+(m-2)\gamma_{121}\},$$

$$\xi_0^{01,02}=\sqrt{(m-1)/2}\{4\gamma_{301}-2\gamma_{202}+(m-2)\gamma_{103}\},$$

$$\xi_0^{01,11}=\sqrt{m-1}\{2\gamma_{310}+(m-2)\gamma_{112}\},$$

$$\xi_0^{20,20}=\left\{4\gamma_{400}+4\gamma_{301}+\gamma_{202}+6(m-2)(2\gamma_{220}+\gamma_{121})+9\binom{m-2}{2}\gamma_{040}\right\}\Big/9,$$

$$\xi_0^{20,02}=\gamma_{220}+2(m-2)\gamma_{121}+\binom{m-2}{2}\gamma_{022},$$

$$\xi_0^{20,11}=\sqrt{2}\left\{2\gamma_{310}+(2m-3)\gamma_{211}+(m-2)(\gamma_{112}+3\gamma_{130})+3\binom{m-2}{2}\gamma_{031}\right\}\Big/3,$$

$$\xi_0^{02,02}=4\gamma_{400}-4\gamma_{301}+(4m-7)\gamma_{202}-2(m-2)\gamma_{103}+\binom{m-2}{2}\gamma_{004},$$

$$\xi_0^{02,11}=\sqrt{2}\left\{2\gamma_{310}+(2m-5)\gamma_{211}+\binom{m-2}{2}\gamma_{013}\right\},$$

$$\xi_0^{11,11}=\left\{4\gamma_{400}+(2m-5)\gamma_{202}+3(2m-3)\gamma_{220}-3(m-4)\gamma_{121}+6\binom{m-2}{2}\gamma_{022}\right\}\Big/3,$$

$$\xi_1^{20,20}=(4\gamma_{400}+4\gamma_{301}+\gamma_{202}-12\gamma_{220}-6\gamma_{121}+9\gamma_{040})/9,$$

$$\xi_1^{20,02}=\gamma_{220}-2\gamma_{121}+\gamma_{022},$$

$$\xi_1^{20,11}=\sqrt{2}(2\gamma_{310}-\gamma_{211}-\gamma_{112}-3\gamma_{130}+3\gamma_{031})/3,$$

$$\xi_1^{02,02}=4\gamma_{400}-4\gamma_{301}-3\gamma_{202}+2\gamma_{103}+\gamma_{004},$$

$$\xi_1^{02,11}=\sqrt{2}(2\gamma_{310}-3\gamma_{211}+\gamma_{013}),$$

$$\xi_1^{11,11}=(4\gamma_{400}-3\gamma_{202}-3\gamma_{220}-\gamma_{103}-3\gamma_{121}+6\gamma_{022})/3,$$

$$\xi_2^{11,11}=(4\gamma_{400}-3\gamma_{202}-9\gamma_{220}-\gamma_{103}+9\gamma_{121})/3,$$

$$\xi_{f_{11}}^{10,10}=(2\gamma_{400}+\gamma_{301}-3\gamma_{220})/3,\quad \xi_{f_{11}}^{10,01}=\gamma_{310}-\gamma_{211},$$

$$\xi_{f_{12}}^{10,20}=\sqrt{m-2}\,(2\gamma_{310}+\gamma_{211}-3\gamma_{130}),\quad \xi_{f_{12}}^{10,02}=\sqrt{m-2}(\gamma_{211}-\gamma_{112}),$$

$$\xi_{f_{13}}^{10,11}=\sqrt{m/2}\,(2\gamma_{301}+\gamma_{202}-3\gamma_{220})/3,$$

$$\xi_{f_{14}}^{10,11}=\sqrt{(m-2)/2}\,(2\gamma_{301}+\gamma_{202}+3\gamma_{220}-6\gamma_{121})/3,$$

$$\xi_{f_{11}}^{01,01}=2\gamma_{400}-\gamma_{301}-\gamma_{202},\quad \xi_{f_{12}}^{01,20}=\sqrt{m-2}(\gamma_{220}-\gamma_{121}),$$

$$\xi_{f_{12}}^{01,02}=\sqrt{m-2}(2\gamma_{301}-\gamma_{202}-\gamma_{103}),\quad \xi_{f_{13}}^{01,11}=\sqrt{2m}(\gamma_{211}-\gamma_{310}),$$

$$\xi_{f_{14}}^{01,11}=\sqrt{2(m-2)}(\gamma_{310}-\gamma_{112}),$$

$$\xi_{f_{22}}^{20,20}=\{4\gamma_{400}+4\gamma_{301}+\gamma_{202}+3(m-4)(2\gamma_{220}+\gamma_{121})-9(m-3)\gamma_{040}\}/9,$$

$$\xi_{f_{22}}^{20,02}=\gamma_{220}+(m-4)\gamma_{121}-(m-3)\gamma_{022},$$

$$\xi_{f_{23}}^{20,11}=\sqrt{m(m-2)/2}(2\gamma_{211}+\gamma_{112}-3\gamma_{130})/3,$$

$$\xi_{f_{24}}^{20,11}=\sqrt{1/2}\{4\gamma_{310}+2(m-3)\gamma_{211}+(m-4)(\gamma_{112}+3\gamma_{130})-6(m-3)\gamma_{031}\}/3,$$

$$\xi_{f_{22}}^{02,02}=4\gamma_{400}-4\gamma_{301}+(2m-7)\gamma_{202}-(m-4)\gamma_{103}-(m-3)\gamma_{004},$$

$$\xi_{f_{23}}^{02,11}=\sqrt{2m(m-2)}(\gamma_{112}-\gamma_{211}),$$

$$\xi_{f_{24}}^{02,11}=\sqrt{2}\{2\gamma_{310}+(m-5)\gamma_{211}-(m-3)\gamma_{013}\},$$

$$\xi_{f_{33}}^{11,11}=\{8\gamma_{400}+2(m-3)(\gamma_{202}+3\gamma_{220})+(m-2)(\gamma_{103}-9\gamma_{121})\}/6,$$

$$\xi_{f_{34}}^{11,11}=\sqrt{m(m-2)/2}\times(2\gamma_{202}+\gamma_{103}-6\gamma_{220}+3\gamma_{121})/6,$$

$$\xi_{f_{44}}^{11,11}=\{8\gamma_{400}+2(m-5)\gamma_{202}+6(m-3)(\gamma_{220}-2\gamma_{022})$$
$$+(m-4)(\gamma_{103}+3\gamma_{121})\}/6. \tag{4.2}$$

(e.g., [12, 14]). Note that a relationship between $\gamma_{p_0p_1p_2}$'s and $\mu_{i_0i_1i_2}$'s of a B-array is given by

$$\gamma_{p_0p_1p_2}=\sum\{p_0!/(i_0^0!\,i_1^0!\,i_2^0!)\}\{p_1!/(i_0^1!\,i_1^1!\,i_2^1!)\}\{p_2!/(i_0^2!\,i_1^2!\,i_2^2!)\}$$
$$\times(-1)^{i_0^1}\delta_{0i_1^1}(-2)^{i_1^2}\mu_{i_0^0+i_0^1+i_0^2\,i_1^0+i_1^1+i_1^2\,i_2^0+i_2^1+i_2^2}, \tag{4.3}$$

where the summation $\sum$ extends over $i_0^x+i_1^x+i_2^x=p_x, 0\leqslant i_y^x\leqslant p_x$ $(x,y=0,1,2)$ and $p_0+p_1+p_2=4$. Thus if T is a $\mathrm{BA}(N,m,3,4;\{\mu_{i_0i_1i_2}\})$, a necessary and sufficient condition for the information matrix $\tilde{M}_T$ to be nonsingular, i.e., for T to be of resolution V, is that H_β $(0\leqslant\beta\leqslant 2)$ and H_f are positive definite matrices (e.g., [12]).

For the rest of this paper, we use the following notations: The summations $\sum_{a_1a_2}^{r}$, $\sum_{c_1c_2}^{*s}$ and $\sum_{u_1u_2;u}^{**t}$ extend, respectively, over all the values of a_1a_2, c_1c_2 and $(u_1u_2;u)$ such that $a_1a_2=00$ if $r=0$; $00,10$ if $r=1$; $00,10,01$ if $r=2$; $00,10,01,20$ if $r=3$; $00,10,01,20,02$ if $r=4$; $00,10,01,20,02,11$ if $r=5$, $c_1c_2=20$ if $s=0$; $20,02$ if $s=1$; $20,02,11$ if $s=2$, and $(u_1u_2;u)=(10;1)$ if $t=0$; $(10;1)$, $(01;1)$ if $t=1$; $(10;1)$, $(01;1)$, $(20;2)$ if $t=2$; $(10;1)$, $(01;1)$, $(20;2)$, $(02;2)$ if $t=3$; $(10;1)$, $(01;1)$, $(20;2)$, $(02;2)$, $(11;3)$ if $t=4$; $(10;1)$, $(01;1)$, $(20;2)$, $(02;2)$, $(11;3)$, $(11;4)$ if $t=5$. Furthermore, $\sum_{b_1b_2}^{-r}$, $\sum_{d_1d_2}^{*-s}$ and $\sum_{v_1v_2;v}^{**-t}$ are, respectively, the summations over all the values of b_1b_2, d_1d_2 and $(v_1v_2;v)$ such that $b_1b_2=10,01,20,02,11$ if $r=0$; $01,20,02,11$ if $r=1$; $20,02,11$ if $r=2$; $02,11$ if $r=3$; 11 if $r=4$; empty if $r=5$, $d_1d_2=02,11$ if $s=0$; 11 if $s=1$; empty if $s=2$, and $(v_1v_2;v)=(01;1)$, $(20;2)$, $(02;2)$, $(11;3)$, $(11;4)$ if $t=0$; $(20;2)$, $(02;2)$, $(11;3)$, $(11;4)$ if $t=1$; $(02;2)$, $(11;3)$, $(11;4)$ if $t=2$; $(11;3)$, $(11;4)$ if $t=3$; $(11;4)$ if $t=4$; empty if $t=5$.

4.2. Structure of MDR algebras

In this section, we consider a 3^m-BFF design of resolution V derived from a BA$(N, m, 3, 4; \{\mu_{i_0 i_1 i_2}\})$, *where* $N > v(m)$. Let

$$F_\beta^{a_1a_2,b_1b_2} = X_{a_1a_2} A_\beta^{(a_1a_2,b_1b_2)\#} X'_{b_1b_2}$$

$$\begin{array}{lll} \text{for } a_1a_2, b_1b_2 = 00, 10, 01, 20, 02, 11 & \text{if } \beta = 0, \\ \qquad 20, 02, 11 & \text{if } \beta = 1, \\ \qquad 11 & \text{if } \beta = 2, \end{array} \tag{4.4a}$$

$$F_{fuv}^{u_1u_2,v_1v_2} = X_{u_1u_2} A_{fuv}^{(u_1u_2,v_1v_2)\#} X'_{v_1v_2}$$

$$\text{for } (u_1u_2; u), (v_1v_2; v) = (10; 1), (01; 1), (20; 2), (02; 2), (11; 3), (11; 4), \tag{4.4b}$$

where $A_\beta^{(a_1a_2,b_1b_2)\#}$'s and $A_{fuv}^{(u_1u_2,v_1v_2)\#}$'s are, respectively, the matrices of size $n(a_1a_2) \times n(b_1b_2)$ and $n(u_1u_2) \times n(v_1v_2)$ given by some linear combinations of the local MDR matrices $A_\alpha^{(a_1a_2,b_1b_2)}$ (see [12, 14]), and $X_{a_1a_2}$'s are $N \times n(a_1a_2)$ submatrices of X_T corresponding to $\Gamma_{a_1a_2}$ for $a_1a_2 = 00, 10, 01, 20, 02, 11$, i.e., $X_T = [X_{00}; X_{10}; X_{01}; X_{20}; X_{02}; X_{11}]$. Here $n(a_1a_2) = 1$ if $a_1a_2 = 00$; m if $a_1a_2 = 10$ or 01; $\binom{m}{2}$ if $a_1a_2 = 20$ or 02; $2\binom{m}{2}$ if $a_1a_2 = 11$. The following are properties of $A_\beta^{(a_1a_2,b_1b_2)}$ and $A_{fuv}^{(u_1u_2,v_1v_2)\#}$.

$$A_0^{(a_1a_2,b_1b_2)\#} = [1/\{n(a_1a_2) \times n(b_1b_2)\}^{1/2}] G_{n(a_1a_2)\times n(b_1b_2)}, \tag{4.5}$$

$$A_0^{(00,00)\#} = [1], \tag{4.6a}$$

$$A_0^{(a_1a_2,a_1a_2)\#} + A_{f11}^{(a_1a_2,a_1a_2)\#} = I_{n(a_1a_2)} \quad \text{for } a_1a_2 = 10, 01, \tag{4.6b}$$

$$A_0^{(a_1a_2,a_1a_2)\#} + A_1^{(a_1a_2,a_1a_2)\#} + A_{f22}^{(a_1a_2,a_1a_2)\#} = I_{n(a_1a_2)} \quad \text{for } a_1a_2 = 20, 02, \tag{4.6c}$$

$$A_0^{(11,11)\#} + A_1^{(11,11)\#} + A_2^{(11,11)\#} + A_{f33}^{(11,11)\#} + A_{f44}^{(11,11)\#} = I_{n(11)}, \tag{4.6d}$$

$$A_\beta^{(a_1a_2,c_1c_2)\#} A_\gamma^{(c_1c_2,b_1b_2)\#} = \delta_{\beta\gamma} A_\beta^{(a_1a_2,b_1b_2)\#}, \tag{4.7a}$$

$$A_{fuw}^{(u_1u_2,w_1w_2)\#} A_{fw^*v}^{(w_1w_2,v_1v_2)\#} = \delta_{ww^*} A_{fuv}^{(u_1u_2,v_1v_2)\#}, \tag{4.7b}$$

$$A_\beta^{(a_1a_2,c_1c_2)\#} A_{fcb}^{(c_1c_2,b_1b_2)\#} = A_{fac}^{(a_1a_2,c_1c_2)\#} A_\beta^{(c_1c_2,b_1b_2)\#} = O_{n(a_1a_2)\times n(b_1b_2)}, \tag{4.7c}$$

$$\operatorname{rank}(A_\beta^{(a_1a_2,b_1b_2)\#}) = \varphi_\beta, \tag{4.8a}$$

$$\operatorname{rank}(A_{fuv}^{(u_1u_2,v_1v_2)\#}) = \varphi_f, \tag{4.8b}$$

(see [12, 14]). From (4.2), (4.3), (4.4a, b) and (4.7a, b, c), we get

$$F_\beta^{a_1a_2,c_1c_2} F_\gamma^{d_1d_2,b_1b_2} = \delta_{\beta\gamma} \xi_\beta^{c_1c_2,d_1d_2} F_\beta^{a_1a_2,b_1b_2}, \tag{4.9a}$$

$$F_{fuw}^{u_1u_2,w_1w_2} F_{fsv}^{s_1s_2,v_1v_2} = \xi_{fws}^{w_1w_2,s_1s_2} F_{fuv}^{u_1u_2,v_1v_2}, \tag{4.9b}$$

$$F_\beta^{a_1a_2,b_1b_2} F_{fuv}^{u_1u_2,v_1v_2} = F_{fuv}^{u_1u_2,v_1v_2} F_\beta^{a_1a_2,b_1b_2} = 0_{N\times N}. \tag{4.9c}$$

Let $\mathscr{B}=[F_\beta^{a_1a_2,b_1b_2}, F_{f_{uv}}^{u_1u_2,v_1v_2} \mid a_1a_2, b_1b_2=00, 10, 01, 20, 02, 11$ if $\beta=0$; $a_1a_2, b_1b_2=20, 02, 11$ if $\beta=1$; $a_1a_2, b_1b_2=11$ if $\beta=2$; $(u_1u_2;u)$, $(v_1v_2;v)=(10;1)$, $(01;1)$, $(20;2)$, $(02;2)$, $(11;3)$, $(11;4),]$, and further let $\mathscr{B}_\beta=[F_\beta^{a_1a_2,b_1b_2} \mid a_1a_2, b_1b_2=00, 10, 01, 20, 02, 11$ if $\beta=0$; $a_1a_2, b_1b_2=20, 02, 11$ if $\beta=1$; $a_1a_2, b_1b_2=11$ if $\beta=2]$ for $0\leqslant\beta\leqslant 2$, and $\mathscr{B}_f=[F_{f_{uv}}^{u_1u_2,v_1v_2} \mid (u_1u_2;u), (v_1v_2;v)=(10;1), (01;1), (20;2), (02;2), (11;3), (11;4),]$. Then using an argument similar to the one used in Theorem 3.1, (4.8a, b) and (4.9a, b, c) show the following (see [18]).

Theorem 4.1. (i) *The matrix algebras $\mathscr{B}_\beta$ $(0\leqslant\beta\leqslant 2)$ and $\mathscr{B}_f$ are minimal two-sided ideals of $\mathscr{B}$, and $\mathscr{B}_\beta\mathscr{B}_\gamma=\delta_{\beta\gamma}\mathscr{B}_\beta$ for $\beta,\gamma=0,1,2$, and $\mathscr{B}_\beta\mathscr{B}_f=\mathscr{B}_f\mathscr{B}_\beta=0$ for $0\leqslant\beta\leqslant 2$.*

(ii) *The matrix algebra $\mathscr{B}$ is decomposed into the direct sum of the ideals $\mathscr{B}_\beta$ $(0\leqslant\beta\leqslant 2)$ and $\mathscr{B}_f$ of $\mathscr{B}$.*

(iii) *The ideals $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_f$ have, respectively, $g_0^{a_1a_2,b_1b_2}$ $g_1^{c_1c_2,d_1d_2}$, $g_2^{11,11}$ and $g_{f_{uv}}^{u_1u_2,v_1v_2}$ as their bases, and they are, respectively, isomorphic to the complete 6×6, 3×3, 1×1 and 6×6 matrix algebras with multiplicities $\varphi_0, \varphi_1, \varphi_2$ and φ_f, where*

$$g_0^{a_1a_2,b_1b_2}=\sum_{p_1p_2}^{5}\xi^0_{b_1b_2,p_1p_2}F_0^{a_1a_2,p_1p_2}, \tag{4.10a}$$

$$g_1^{c_1c_2,d_1d_2}=\sum_{q_1q_2}^{*2}\xi^1_{d_1d_2,q_1q_2}F_1^{c_1c_2,q_1q_2}, \tag{4.10b}$$

$$g_2^{11,11}=\xi^2_{11,11}F_2^{11,11}, \tag{4.10c}$$

$$g_{f_{uv}}^{u_1u_2,v_1v_2}=\sum_{w_1w_2;w}^{**5}\xi^{f_{uw}}_{v_1v_2,w_1w_2}F_{f_{uw}}^{u_1u_2,w_1w_2}, \tag{4.10d}$$

$$H_\beta^{-1}=\|\xi^\beta_{a_1a_2,b_1b_2}\| \quad \textit{for } 0\leqslant\beta\leqslant 2,$$

$$H_f^{-1}=\|\xi^{f_{uv}}_{u_1u_2,v_1v_2}\|.$$

Let

$$\tilde{P}_0^0+\tilde{P}_0^1+\cdots+\tilde{P}_0^r$$
$$=\sum_{a_1a_2}^{r}\left[g_0^{a_1a_2,a_1a_2}+\sum_{q_1q_2}^{-r}\left\{\sum_{b_1b_2}^{r}\tau_0^{a_1a_2,b_1b_2}(r)\,\xi_0^{q_1q_2,b_1b_2}\right\}g_0^{a_1a_2,q_1q_2}\right]$$

for $0\leqslant r\leqslant 5$,

$$\tilde{P}_1^0+\tilde{P}_1^1+\cdots+\tilde{P}_1^s$$
$$=\sum_{c_1c_2}^{*s}\left[g_1^{c_1c_2,c_1c_2}+\sum_{q_1q_2}^{*-s}\left\{\sum_{d_1d_2}^{*s}\tau_1^{c_1c_2,d_1d_2}(s)\,\xi_1^{q_1q_2,d_1d_2}\right\}g_1^{c_1c_2,q_1q_2}\right]$$

for $0\leqslant s\leqslant 2$,

$$\tilde{P}_2^0=g_2^{11,11},$$

$$\tilde{P}_f^0+\tilde{P}_f^1+\cdots+\tilde{P}_f^t$$

$$= \sum_{u_1 u_2 ; u}^{**t} \left[g_{f_{uu}}^{u_1 u_2, d_1 u_2} + \sum_{q_1 q_2 ; q}^{*-t} \left\{ \sum_{v_1 v_2 ; v}^{**t} \tau_{f_{uv}}^{u_1 u_2, v_1 v_2}(t) \xi_{f_{qv}}^{q_1 q_2, v_1 v_2} \right\} g_{f_{uq}}^{u_1 u_2, q_1 q_2} \right]$$

for $0 \leqslant t \leqslant 5$,

where

$$H_0(r)^{-1} = \| \tau_0^{a_1 a_2, b_1 b_2}(r) \| \quad \text{for } 0 \leqslant r \leqslant 5,$$

$$H_1(s)^{-1} = \| \tau_1^{c_1 c_2, d_1 d_2}(s) \| \quad \text{for } 0 \leqslant s \leqslant 2,$$

$$H_2(0)^{-1} = \| \tau_2^{11, 11}(0) \|,$$

$$H_f(t)^{-1} = \| \tau_{f_{uv}}^{u_1 u_2, v_1 v_2}(t) \| \quad \text{for } 0 \leqslant t \leqslant 5.$$

Here, $H_\beta(r)$ ($0 \leqslant r \leqslant 5$ if $\beta = 0$; $0 \leqslant r \leqslant 2$ if $\beta = 1$; $r = 0$ if $\beta = 2$) and H_f ($0 \leqslant t \leqslant 5$) are the $(r+1) \times (r+1)$ and $(t+1) \times (t+1)$ submatrices of H_β and H_f, respectively, which include the first $(r+1)$ rows and columns of H_β and the first $(t+1)$ rows and columns of H_f. The following results due to [18] can be proved along lines similar to those used in Section 3.2.

$$\tilde{P}_0^r = \sum_{a_1 a_2}^{r} \sum_{b_1 b_2}^{r} \tau_0^{a_1 a_2, b_1 b_2}(r) F_0^{a_1 a_2, b_1 b_2}$$

$$- \sum_{i_1 i_2}^{r-1} \sum_{j_1 j_2}^{r-1} \tau_0^{i_1 i_2, j_1 j_2}(r-1) F_0^{i_1 i_2, j_1 j_2} \quad \text{for } 0 \leqslant r \leqslant 5,$$

$$\tilde{P}_1^s = \sum_{c_1 c_2}^{*s} \sum_{d_1 d_2}^{*s} \tau_1^{c_1 c_2, d_1 d_2}(s) F_1^{c_1 c_2, d_1 d_2}$$

$$- \sum_{i_1 i_2}^{*s-1} \sum_{j_1 j_2}^{*s-1} \tau_1^{i_1 i_2, j_1 j_2}(s-1) F_1^{i_1 i_2, j_1 j_2} \quad \text{for } 0 \leqslant s \leqslant 2,$$

$$\tilde{P}_2^0 = \tau_2^{11, 11}(0) F_2^{11, 11} = (1/\xi_2^{11, 11}) F_2^{11, 11},$$

$$\tilde{P}_f^t = \sum_{u_1 u_2 ; u}^{**t} \sum_{v_1 v_2 ; v}^{**t} \tau_{f_{uv}}^{u_1 u_2, v_1 v_2}(t) F_{f_{uv}}^{u_1 u_2, v_1 v_2}$$

$$- \sum_{p_1 p_2 ; p}^{**-t} \sum_{q_1 q_2 ; q}^{**t-1} \tau_{f_{pq}}^{p_1 p_2, q_1 q_2}(t-1) F_{f_{pq}}^{p_1 p_2, q_1 q_2} \quad \text{for } 0 \leqslant t \leqslant 5,$$

where $\tau_\beta^{a_1 a_2, b_1 b_2}(-1) = \tau_{f_{uv}}^{u_1 u_2, v_1 v_2}(-1) = 0$. The formulas (4.8a, b), (4.9a, b, c) and (4.10a, b, c, d) yield the following (see [18]).

Theorem 4.2. (i) *The* $\tilde{P}_\beta^r$ ($0 \leqslant r \leqslant 5$ *if* $\beta = 0$; $0 \leqslant r \leqslant 2$ *if* $\beta = 1$; $r = 0$ *if* $\beta = 2$), $\tilde{P}_f^t$ ($0 \leqslant t \leqslant 5$) *and* $\tilde{P}_e$ *are symmetric and mutually orthogonal idempotent matrices, where*

$$\tilde{P}_e = I_N - \left\{ \sum_{r=0}^{5} \tilde{P}_0^r + \sum_{s=0}^{2} \tilde{P}_1^s + \tilde{P}_2^0 + \sum_{t=0}^{5} \tilde{P}_f^t \right\}.$$

(ii) $\operatorname{rank}(\tilde{P}_\beta^r)=\varphi_\beta$ *for* $0\leqslant r\leqslant 5$ *if* $\beta=0$, $0\leqslant r\leqslant 2$ *if* $\beta=1$, $r=0$ *if* $\beta=2$,

$\operatorname{rank}(\tilde{P}_f^t)=\varphi_f$ *for* $0\leqslant t\leqslant 5$,

$\operatorname{rank}(\tilde{P}_e)=N-v(m)$.

4.3. ANOVA of 3^m-BFF designs of resolution V

We now consider the ANOVA of a 3^m-BFF design of resolution V derived from a $\mathrm{BA}(N,m,3,4;\{\mu_{i_0i_1i_2}\})$, where $N>v(m)$. From (4.6a, b, c, d), the linear model (4.1) is rewritten as

$$\tilde{y}(T)=\sum_{a_1a_2}^{5} X_{a_1a_2}A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$$

$$+\sum_{c_1c_2}^{*2} X_{c_1c_2}A_1^{(c_1c_2,c_1c_2)\#}\Gamma_{c_1c_2}+X_{11}A_2^{(11,11)\#}\Gamma_{11}$$

$$+\sum_{u_1u_2;u}^{**5} X_{u_1u_2}A_{fuu}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}+\tilde{e}_T.$$

The formulas (4.5), (4.7a, b, c) and (4.8a, b) imply that (i) every element of the vectors $A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$ ($a_1a_2=00,10,01,20,02,11$) represents the average of these effects, (ii) the elements of $A_\beta^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$ ($\beta\neq 0$) and $A_{faa}^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$ represent the contrasts among corresponding effects, (iii) any two different contrasts are orthogonal, and (iv) there exists φ_β ($0\leqslant\beta\leqslant 2$) and φ_f independent parametric functions of $\Gamma_{a_1a_2}$ in $A_\beta^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$ and $\Gamma_{u_1u_2}$ in $A_{fuu}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}$, respectively.

Let $\tilde{H}_0^r$, $\tilde{H}_1^s$, $\tilde{H}_2^0$ and $\tilde{H}_f^t$ be the hypotheses that

$$A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}=\mathbf{0}_{n(a_1a_2)},\qquad A_1^{(c_1c_2,c_1c_2)\#}\Gamma_{c_1c_2}=\mathbf{0}_{n(c_1c_2)},$$

$$A_2^{(11,11)\#}\Gamma_{11}=\mathbf{0}_{n(11)},\qquad A_{fuu}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}=\mathbf{0}_{n(u_1u_2)},$$

respectively, where $r=0,1,2,3,4,5$ correspond to $a_1a_2=00,10,01,20,02,11$, respectively, $s=0,1,2$ correspond to $c_1c_2=20,02,11$, respectively, and $t=0,1,2,3,4,5$ correspond to $(u_1u_2;u)=(10;1)$, $(01;1)$, $(20;2)$, $(02;2)$, $(11;3)$, $(11;4)$, respectively. Further let $\tilde{H}_0^{i\cdots 5}=\bigcap_{r=i}^{5}\tilde{H}_0^r$ for $0\leqslant i\leqslant 5$, $\tilde{H}_1^{j\cdots 2}=\bigcap_{s=j}^{2}\tilde{H}_1^s$ for $0\leqslant j\leqslant 2$, and $\tilde{H}_f^{k\cdots 5}=\bigcap_{t=k}^{5}\tilde{H}_f^t$ for $0\leqslant k\leqslant 5$. First we wish to test the hypotheses $\tilde{H}_0^5$, $\tilde{H}_1^2$, $\tilde{H}_2^0$ and $\tilde{H}_f^5$ against $\tilde{K}_0^5$, $\tilde{K}_1^2$, $\tilde{K}_2^0$ and $\tilde{K}_f^5$, respectively, where $\tilde{K}_0^5$, $\tilde{K}_1^2$, $\tilde{K}_2^0$ and $\tilde{K}_f^5$ are the hypotheses that

$$A_0^{(11,11)\#}\Gamma_{11}\neq\mathbf{0}_{n(11)},\qquad A_1^{(11,11)\#}\Gamma_{11}\neq\mathbf{0}_{n(11)},$$

$$A_2^{(11,11)\#}\Gamma_{11}\neq\mathbf{0}_{n(11)},\qquad A_{f44}^{(11,11)\#}\Gamma_{11}\neq\mathbf{0}_{n(11)},$$

respectively. Next if $\tilde{H}_0^5$ ($\tilde{H}_1^2$ or $\tilde{H}_f^5$) is accepted, then we consider testing of hypotheses $\tilde{H}_0^{45}$ ($\tilde{H}_1^{12}$ or $\tilde{H}_f^{45}$) against $\tilde{H}_0^5$ ($\tilde{H}_1^2$ or $\tilde{H}_f^5$), and so on. We note that $\tilde{H}_0^5 \cap \tilde{H}_1^2 \cap \tilde{H}_2^0 \cap \tilde{H}_f^{45}$ and $\tilde{H}_1^2 \cap \tilde{H}_2^0 \cap \tilde{H}_f^{45}$ mean that $\Gamma_{11} = \mathbf{0}_{n(11)}$ and $\Gamma_{11} = c_{11}\mathbf{1}_{n(11)}$, respectively, where c_{11} is a constant. Thus we accept $H_0^{i\cdots 5}$($H_1^{j\cdots 2}$ or $H_f^{k\cdots 5}$) against K_0^5 ($\tilde{K}_1^2$ or $\tilde{K}_f^5$) only if all the tests $\tilde{H}_0^5$ against $\tilde{K}_0^5, \ldots,$ and $\tilde{H}_0^{i\cdots 5}$ against $\tilde{H}_0^{i+1\cdots 5}$ ($\tilde{H}_1^2$ against $\tilde{K}_1^2, \ldots,$ and $\tilde{H}_1^{j\cdots 2}$ against $\tilde{H}_1^{j+1\cdots 2}$ or $\tilde{H}_f^5$ against $\tilde{K}_f^5, \ldots,$ and $\tilde{H}_f^{k\cdots 5}$ against $\tilde{H}_f^{k+1\cdots 5}$) are not significant for $0 \leqslant i \leqslant 5$ ($0 \leqslant j \leqslant 2$ or $0 \leqslant k \leqslant 5$).

Let

$$\tilde{\mathscr{R}}_0^r = \mathscr{R}_{(L_0^{r-1})^\perp}(L_0^r) \quad \text{for } 0 \leqslant r \leqslant 5,$$

$$\tilde{\mathscr{R}}_1^s = \mathscr{R}_{(L_1^{s-1})^\perp}(L_1^s) \quad \text{for } 0 \leqslant s \leqslant 2, \qquad \tilde{\mathscr{R}}_2^0 = \mathscr{R}(X_{11}A_2^{(11,11)\#}),$$

$$\tilde{\mathscr{R}}_f^t = \mathscr{R}_{(L_f^{t-1})^\perp}(L_f^t) \quad \text{for } 0 \leqslant t \leqslant 5 \quad \text{and} \quad \tilde{\mathscr{R}}_e = \mathscr{R}_{X_T^\perp},$$

where

$$L_\beta^{-1} = [\mathbf{0}_N] \quad \text{for } 0 \leqslant \beta \leqslant 2, \; L_f^{-1} = [\mathbf{0}_N],$$

$$L_0^r = [X_{00}A_0^{(00,00)\#}; \ldots; X_{a_1a_2}A_0^{(a_1a_2,a_1a_2)\#}] \quad \text{for } 0 \leqslant r \leqslant 5,$$

$$L_1^s = [X_{20}A_1^{(20,20)\#}; \ldots; X_{c_1c_2}A_1^{(c_1c_2,c_1c_2)\#}] \quad \text{for } 0 \leqslant s \leqslant 2,$$

$$L_f^t = [X_{10}A_{f_{11}}^{(10,10)\#}; \ldots; X_{u_1u_2}A_{f_{uu}}^{(u_1u_2,u_1u_2)\#}] \quad \text{for } 0 \leqslant t \leqslant 5.$$

Then from Theorem 4.2, we get

$$\mathscr{R}^N = \tilde{\mathscr{R}}_0^0 \oplus \tilde{\mathscr{R}}_0^1 \oplus \cdots \oplus \tilde{\mathscr{R}}_0^5 \oplus \tilde{\mathscr{R}}_1^0 \oplus \tilde{\mathscr{R}}_1^1 \oplus \tilde{\mathscr{R}}_1^2 \oplus \tilde{\mathscr{R}}_2^0 \oplus \tilde{\mathscr{R}}_f^0 \oplus \tilde{\mathscr{R}}_f^1 \oplus \cdots \oplus \tilde{\mathscr{R}}_f^5 \oplus \tilde{\mathscr{R}}_e. \tag{4.11}$$

Note that $\tilde{P}_0^r$ $(0 \leqslant r \leqslant 5)$, $\tilde{P}_1^s$ $(0 \leqslant s \leqslant 2)$, $\tilde{P}_2^0$, $\tilde{P}_f^t (0 \leqslant t \leqslant 5)$ and $\tilde{P}_e$ are projections of $\mathscr{R}^N$ onto $\tilde{\mathscr{R}}_0^r$, $\tilde{\mathscr{R}}_1^s$, $\tilde{\mathscr{R}}_2^0$, $\tilde{\mathscr{R}}_f^t$ and $\tilde{\mathscr{R}}_e$, respectively.

Consider $\tilde{S}_0^r (0 \leqslant r \leqslant 5)$, $\tilde{S}_1^s (0 \leqslant s \leqslant 2)$, $\tilde{S}_2^0$, $\tilde{S}_f^t (0 \leqslant t \leqslant 5)$ and $\tilde{S}_e$ corresponding, respectively, to the SS's due to $A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$ $(a_1a_2 = 00, 10, 01, 20, 02, 11)$, $A_1^{(c_1c_2,c_1c_2)\#}\Gamma_{c_1c_2}$ $(c_1c_2 = 20, 02, 11)$, $A_2^{(11,11)\#}\Gamma_{11}$, $A_{f_{uu}}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}$ $((u_1u_2; u) = (10; 1)$, $(01; 1)$, $(20; 2)$, $(02; 2)$, $(11; 3)$, $(11; 4))$ and an error, where

$$\tilde{S}_0^r = \tilde{y}(T)'\tilde{P}_0^r\tilde{y}(T) \quad \text{for } 0 \leqslant r \leqslant 5,$$

$$\tilde{S}_1^s = \tilde{y}(T)'\tilde{P}_1^s\tilde{y}(T) \quad \text{for } 0 \leqslant s \leqslant 2, \qquad \tilde{S}_2^0 = \tilde{y}(T)'\tilde{P}_2^0\tilde{y}(T),$$

$$\tilde{S}_f^t = \tilde{y}(T)'\tilde{P}_f^t\tilde{y}(T) \quad \text{for } 0 \leqslant t \leqslant 5 \quad \text{and} \quad \tilde{S}_e = \tilde{y}(T)'\tilde{P}_e\tilde{y}(T).$$

Then we have the following (see [18]).

Theorem 4.3.

$$\tilde{y}(T)'\tilde{y}(T) = \sum_{r=0}^{5} \tilde{S}_0^r + \sum_{s=0}^{2} \tilde{S}_1^s + \tilde{S}_2^0 + \sum_{t=0}^{5} \tilde{S}_f^t + \tilde{S}_e.$$

Theorem 4.4. *An unbiased estimator of* $\tilde{\sigma}^2$ *is given by* $\hat{\tilde{\sigma}}^2 = \tilde{S}_e / \{N - v(m)\}$.

Let

$$c_0^r(a_1 a_2, b_1 b_2)$$

$$= \begin{cases} \xi_0^{a_1 a_2, 00} \xi_0^{00, b_1 b_2} / \xi_0^{00,00} & \text{if } r=0, \\ \left\{ \det \begin{bmatrix} H_0(r-1) & C_0^{12} \\ \boldsymbol{x}_0(a_1 a_2)' & x_0(a_1 a_2) \end{bmatrix} \right\} \left\{ \det \begin{bmatrix} H_0(r-1) & \boldsymbol{y}_0(b_1 b_2) \\ C_0^{21\prime} & y_0(b_1 b_2) \end{bmatrix} \right\} \Big/ & \\ \quad [\{\det(H_0(r))\} \{\det(H_0(r-1))\}] & \text{if } 1 \leqslant r \leqslant 5, \end{cases}$$

$$c_1^s(c_1 c_2, d_1 d_2)$$

$$= \begin{cases} \xi_1^{c_1 c_2, 20} \xi_1^{20, d_1 d_2} / \xi_1^{20,20} & \text{if } s=0, \\ \left\{ \det \begin{bmatrix} H_1(s-1) & C_1^{12} \\ \boldsymbol{x}_1(c_1 c_2)' & x_1(c_1 c_2) \end{bmatrix} \right\} \left\{ \det \begin{bmatrix} H_1(s-1) & \boldsymbol{y}_1(d_1 d_2) \\ C_1^{21\prime} & y_1(d_1 d_2) \end{bmatrix} \right\} \Big/ & \\ \quad [\{\det(H_1(s))\} \{\det(H_1(s-1))\}] & \text{if } 1 \leqslant s \leqslant 2, \end{cases}$$

$$c_2^0(11, 11) = \xi_2^{11,11},$$

$$c_f^t((u_1 u_2; u), (v_1 v_2; v))$$

$$= \begin{cases} \xi_{f u 1}^{u_1 u_2, 10} \xi_{f 1 v}^{10, v_1 v_2} / \xi_{f 11}^{10,10} & \text{if } t=0, \\ \left\{ \det \begin{bmatrix} H_f(t-1) & C_f^{12} \\ \boldsymbol{x}_f(u_1 u_2; u)' & x_f(u_1 u_2; u) \end{bmatrix} \right\} \left\{ \det \begin{bmatrix} H_f(t-1) & \boldsymbol{y}_f(v_1 v_2; v) \\ C_f^{21\prime} & y_f(v_1 v_2; v) \end{bmatrix} \right\} \Big/ & \\ \quad [\{\det(H_f(t))\} \{\det(H_f(t-1))\}] & \text{if } 1 \leqslant t \leqslant 5, \end{cases}$$

where C_0^{12}, C_1^{12} and C_f^{12} (or C_0^{21}, C_1^{21} and C_f^{21}) are the $r \times 1$, $s \times 1$ and $t \times 1$ vectors, respectively, whose elements correspond, respectively, to the first r, s and t ones of the $(r+1)$st, $(s+1)$st and $(t+1)$st column (or row) vectors of H_0, H_1 and H_f; where $\boldsymbol{x}_0(a_1 a_2)$, $\boldsymbol{x}_1(c_1 c_2)$ and $\boldsymbol{x}_f(u_1 u_2; u)$ (or $\boldsymbol{y}_0(b_1 b_2)$, $\boldsymbol{y}_1(d_1 d_2)$ and $\boldsymbol{y}_f(v_1 v_2; v)$) are the $r \times 1, s \times 1$ and $t \times 1$ vectors, respectively, whose elements are, respectively, the first r, s and t ones corresponding to $a_1 a_2$th, $c_1 c_2$th and $(u_1 u_2; u)$th row (or $b_1 b_2$th, $d_1 d_2$th and $(v_1 v_2; v)$th column) vectors of H_0, H_1 and H_f; where $x_0(a_1 a_2), x_1(c_1 c_2)$ and $x_f(u_1 u_2; u)$ (or $y_0(b_1 b_2)$, $y_1(d_1 d_2)$ and $x_f(v_1 v_2; v)$) are, respectively, the $a_1 a_2$th row and $(r+1)$st column element of H_0, the $c_1 c_2$th row and $(s+1)$st column one of H_1 and the $(u_1 u_2; u)$th row and $(t+1)$st column one of H_f (or the $(r+1)$st row and $b_1 b_2$th column one of H_0, the $(s+1)$st row and $d_1 d_2$th column one of H_1 and the $(t+1)$st row and $(v_1 v_2; v)$th column one of H_f). Then using an argument similar to the one used in Theorem 3.5, the following theorem can easily be proved (see [18]).

Theorem 4.5. *The noncentrality parameters of the quadratic forms* $\tilde{y}(T)'\tilde{P}_0^r\tilde{y}(T)/\tilde{\sigma}^2$ $(0\leqslant r\leqslant 5)$, $\tilde{y}(T)'\tilde{P}_1^s\tilde{y}(T)/\tilde{\sigma}^2$ $(0\leqslant s\leqslant 2)$, $\tilde{y}(T)'\tilde{P}_2^0\tilde{y}(T)/\tilde{\sigma}^2$ *and* $\tilde{y}(T)'\tilde{P}_f^t\tilde{y}(T)/\tilde{\sigma}^2$ $(0\leqslant t\leqslant 5)$ *are given by*

$$\tilde{\lambda}_0^r/\tilde{\sigma}^2=\sum_{a_1a_2}^{(r)}\sum_{b_1b_2}^{(r)}\{c_0^r(a_1a_2,b_1b_2)/\tilde{\sigma}^2\}\Gamma'_{a_1a_2}A_0^{(a_1a_2,b_1b_2)\#}\Gamma_{b_1b_2},$$

$$\tilde{\lambda}_1^s/\tilde{\sigma}^2=\sum_{c_1c_2}^{*(s)}\sum_{d_1d_2}^{*(s)}\{c_1^s(c_1c_2,d_1d_2)/\tilde{\sigma}^2\}\Gamma'_{c_1c_2}A_1^{(c_1c_2,d_1d_2)\#}\Gamma_{d_1d_2},$$

$$\tilde{\lambda}_2^0/\tilde{\sigma}^2=\{c_2^0(11,11)/\tilde{\sigma}^2\}\Gamma'_{11}A_2^{(11,11)\#}\Gamma_{11},$$

$$\tilde{\lambda}_f^t/\tilde{\sigma}^2=\sum_{u_1u_2;u}^{**(t)}\sum_{v_1v_2;v}^{**(t)}\{c_f^t((u_1u_2;u),(v_1v_2;v))/\tilde{\sigma}^2\}\Gamma'_{u_1u_2}A_{fuv}^{(u_1u_2,v_1v_2)\#}\Gamma_{v_1v_2},$$

respectively, where $\sum_{a_1a_2}^{(0)}$, $\sum_{a_1a_2}^{(r)}$ $(1\leqslant r\leqslant 5)$, $\sum_{c_1c_2}^{*(0)}$, $\sum_{c_1c_2}^{*(s)}$ $(s=1,2)$, $\sum_{u_1u_2;u}^{**(0)}$ *and* $\sum_{u_1u_2;u}^{**(t)}$ $(1\leqslant t\leqslant 5)$ *mean* $\sum_{a_1a_2}^{5}$, $\sum_{a_1a_2}^{-(r-1)}$ $(1\leqslant r\leqslant 5)$, $\sum_{c_1c_2}^{*2}$, $\sum_{c_1c_2}^{*-(s-1)}$ $(s=1,2)$, $\sum_{u_1u_2;u}^{**5}$ *and* $\sum_{u_1u_2;u}^{**-(t-1)}$ $(1\leqslant t\leqslant 5)$, *respectively.*

Some of these results are summarized in Table 3.

Table 3
ANOVA of 3^m-BFF designs of resolution V

Source	SS	d.f.	Noncentrality parameters
$A_0^{(00,00)\#}\Gamma_{00}$	$\tilde{S}_0^0$	φ_0	$\tilde{\lambda}_0^0/\tilde{\sigma}^2$
$A_0^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_0^1$	φ_0	$\tilde{\lambda}_0^1/\tilde{\sigma}^2$
$A_0^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_0^2$	φ_0	$\tilde{\lambda}_0^2/\tilde{\sigma}^2$
$A_0^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_0^3$	φ_0	$\tilde{\lambda}_0^3/\tilde{\sigma}^2$
$A_0^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_0^4$	φ_0	$\tilde{\lambda}_0^4/\tilde{\sigma}^2$
$A_0^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_0^5$	φ_0	$\tilde{\lambda}_0^5/\tilde{\sigma}^2$
$A_1^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_1^0$	φ_1	$\tilde{\lambda}_1^0/\tilde{\sigma}^2$
$A_1^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_1^1$	φ_1	$\tilde{\lambda}_1^1/\tilde{\sigma}^2$
$A_1^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_1^2$	φ_1	$\tilde{\lambda}_1^2/\tilde{\sigma}^2$
$A_2^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_2^0$	φ_2	$\tilde{\lambda}_2^0/\tilde{\sigma}^2$
$A_{f11}^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_f^0$	φ_f	$\tilde{\lambda}_f^0/\tilde{\sigma}^2$
$A_{f11}^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_f^1$	φ_f	$\tilde{\lambda}_f^1/\tilde{\sigma}^2$
$A_{f22}^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_f^2$	φ_f	$\tilde{\lambda}_f^2/\tilde{\sigma}^2$
$A_{f22}^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_f^3$	φ_f	$\tilde{\lambda}_f^3/\tilde{\sigma}^2$
$A_{f33}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^4$	φ_f	$\tilde{\lambda}_f^4/\tilde{\sigma}^2$
$A_{f44}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^5$	φ_f	$\tilde{\lambda}_f^5/\tilde{\sigma}^2$
Error	$\tilde{S}_e$	$N-v(m)$	
Total	$\tilde{y}(T)'\tilde{y}(T)$	N	

Note that from Theorem 4.5, $\tilde{H}_0^{i\cdots 5}$ against $\tilde{K}_0^5$ $(0\leqslant i\leqslant 5)$, $\tilde{H}_1^{j\cdots 2}$ against $\tilde{K}_1^2$ $(0\leqslant j\leqslant 2)$, $\tilde{H}_2^0$ against $\tilde{K}_2^0$, and $\tilde{H}_f^{k\cdots 5}$ against $\tilde{K}_f^5$ $(0\leqslant k\leqslant 5)$ mean that $\tilde{\lambda}_0^i=0$, $\tilde{\lambda}_1^j=0$, $\tilde{\lambda}_2^0=0$ and $\tilde{\lambda}_f^k=0$, respectively, and vice versa.

We shall consider testing of hypotheses $\tilde{H}_0^{i\cdots 5}$ against $\tilde{K}_0^5$ $(0\leqslant i\leqslant 5)$, $\tilde{H}_1^{j\cdots 2}$ against $\tilde{K}_1^2$ $(0\leqslant j\leqslant 2)$, $\tilde{H}_2^0$ against $\tilde{K}_2^0$, and $\tilde{H}_f^{k\cdots 5}$ against $\tilde{K}_f^5$ $(0\leqslant k\leqslant 5)$. The test statistics for the nested method

$$\frac{\tilde{S}_0^5/\varphi_0}{\tilde{S}_e/\{N-v(m)\}}\quad (=\tilde{F}_0^5,\ \text{say}),\qquad \frac{\tilde{S}_0^4/\varphi_0}{(\tilde{S}_e+\tilde{S}_0^5)/\{N-v(m)+\varphi_0\}}\quad (=\tilde{F}_0^{45},\ \text{say}),$$

$$\ldots,\ \text{and}\quad \frac{\tilde{S}_0^i/\varphi_0}{(\tilde{S}_e+\tilde{S}_0^{i+1}+\cdots+\tilde{S}_0^5)/\{N-v(m)+(5-i)\varphi_0)}\quad (=\tilde{F}_0^{i\cdots 5},\ \text{say}),$$

$$\frac{\tilde{S}_1^2/\varphi_1}{\tilde{S}_e/\{N-v(m)\}}\quad (=\tilde{F}_1^2,\ \text{say}),\qquad \frac{\tilde{S}_1^1/\varphi_1}{(\tilde{S}_e+\tilde{S}_1^2)/\{N-v(m)+\varphi_1\}}\quad (=\tilde{F}_1^{12},\ \text{say}),$$

$$\ldots,\ \text{and}\quad \frac{\tilde{S}_1^j/\varphi_1}{(\tilde{S}_e+\tilde{S}_1^{j+1}+\cdots+\tilde{S}_1^2)/\{N-v(m)+(2-j)\varphi_1\}}\quad (=\tilde{F}_1^{j\cdots 2},\ \text{say}),$$

$$\frac{\tilde{S}_2^0/\varphi_2}{\tilde{S}_e/\{N-v(m)\}}\quad (=\tilde{F}_2^0,\ \text{say}),$$

$$\frac{\tilde{S}_f^5/\varphi_f}{\tilde{S}_e/\{N-v(m)\}}\quad (=\tilde{F}_f^5,\ \text{say}),\qquad \frac{\tilde{S}_f^4/\varphi_f}{(\tilde{S}_e+\tilde{S}_f^5)/N-v(m)+\varphi_f\}}\quad (=\tilde{F}_f^{45},\ \text{say}),$$

$$\ldots,\ \text{and}\quad \frac{\tilde{S}_f^k/\varphi_f}{(\tilde{S}_e+\tilde{S}_f^{k+1}+\cdots+\tilde{S}_f^5)/\{N-v(m)+(5-k)\varphi_f\}}\quad (=\tilde{F}_f^{k\cdots 5},\ \text{say})$$

have F distributions. The nested procedure is continued until a significant test is obtained for each case. We remark that $\tilde{F}_0^{i\cdots 5}$ $(0\leqslant i\leqslant 5)$, $\tilde{F}_1^{j\cdots 2}$ $(0\leqslant j\leqslant 2)$, $\tilde{F}_2^0$ and $\tilde{F}_f^{k\cdots 5}$ $(0\leqslant k\leqslant 5)$ are noncentral F distributions with φ_0 and $\{N-v(m)+(5-i)\varphi_0\}$ d.f., φ_1 and $\{N-v(m)+(2-j)\varphi_1\}$ d.f., φ_2 and $\{N-v(m)\}$ d.f., and φ_f and $(N-v(m)+(5-k)\varphi_f\}$ d.f., and noncentrality parameters $\tilde{\lambda}_0^i/\tilde{\sigma}^2$, $\tilde{\lambda}_1^j/\tilde{\sigma}^2$, $\tilde{\lambda}_2^0/\tilde{\sigma}^2$ and $\tilde{\lambda}_f^k/\tilde{\sigma}^2$ if $\tilde{H}_0^{i\cdots 5}$, $\tilde{H}_1^{j\cdots 2}$, $\tilde{H}_2^0$ and $\tilde{H}_f^{k\cdots 5}$ are false, respectively.

4.4. ANOVA of some 3^m-BFF designs of resolution IV

Under the assumption that three-factor and higher order interactions are negligible, if $(\Gamma'_{10},\Gamma'_{01})'$ (or $(\Gamma'_{00},\Gamma'_{10},\Gamma'_{01})'$) is estimable, a 3^m-FF design is said to be of resolution IV. When $N\geqslant v(m)$, there may exist a 3^m-BFF design of resolution V derived from a B-array. While for $N=v(m)$, there is no d.f. due to error. Thus we restrict the total number of assemblies as $N\leqslant v(m)$. Some 3^m-BFF designs of resolution IV derived from BA$(N,m,3,4;\ \{\mu_{i_0i_1i_2}\})$ were obtained by Kuwada [15]. In this section, we consider the following two cases.

(A) $\det(H_\beta)\neq 0$ for $\beta=0,1$, $\det(H_f)\neq 0$ and $\det(H_2)=0$.

(B) $\det(H_0)\neq 0$, $\det(H_f)\neq 0$ and $\det(H_1)=\det(H_2)=0$.

Note that $\det(H_2)=0$, and $\det(H_2)=0$ and $\det(H_1)=0$ imply that $\mu_{211}=\mu_{121}=\mu_{112}=0$, and $\mu_{211}=\mu_{121}=\mu_{112}=0$ and at least one of μ_{220}, μ_{202} and μ_{022} is zero, respectively. For a design satisfying Condition (A) (or (B)), the linear model (4.1) is rewritten as

$$\tilde{y}^*(T)=\sum_{a_1a_2}^{5} X_{a_1a_2} A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}+\sum_{c_1c_2}^{*2} X_{c_1c_2} A_1^{(c_1c_2,c_1c_2)\#}\Gamma_{c_1c_2}$$

$$+\sum_{u_1u_2;u}^{**5} X_{u_1u_2} A_{f_{uu}}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}+\tilde{e}_T^*$$

$$\Big(\text{or } \tilde{y}^{**}(T)=\sum_{a_1a_2}^{5} X_{a_1a_2} A_0^{(a_1a_2,a_1a_2)\#}\Gamma_{a_1a_2}$$

$$+\sum_{u_1u_2;u}^{**5} X_{u_1u_2} A_{f_{uu}}^{(u_1u_2,u_1u_2)\#}\Gamma_{u_1u_2}+\tilde{e}_T^{**}\Big),$$

where $\tilde{e}_T^*$ (or $\tilde{e}_T^{**}$) is an error vector distributed as $\mathcal{N}(\mathbf{0}_N, \tilde{\sigma}^{*2}I_N)$ (or $\mathcal{N}(\mathbf{0}_N, \tilde{\sigma}^{**2}I_N)$). Note that a design satisfying Condition (A) (or (B)) is of resolution IV and $(\Gamma'_{00}, \Gamma'_{10}, \Gamma'_{01}, \Gamma'_{20}, \Gamma'_{02})'$ (or $(\Gamma'_{00}, \Gamma'_{10}, \Gamma'_{01})$) and some linear combinations in Γ_{11} (or Γ_{20}, Γ_{02} and Γ_{11}) such that $A_0^{(11,11)\#}\Gamma_{11}$, $A_1^{(11,11)\#}\Gamma_{11}$, $A_{f_{33}}^{(11,11)\#}\Gamma_{11}$ and $A_{f_{44}}^{(11,11)\#}\Gamma_{11}$ (or $A_0^{(20,20)\#}\Gamma_{20}$, $A_{f_{22}}^{(20,20)\#}\Gamma_{20}$, $A_0^{(02,02)\#}\Gamma_{02}$, $A_{f_{22}}^{(02,02)\#}\Gamma_{02}$, $A_0^{(11,11)\#}\Gamma_{11}$, $A_{f_{33}}^{(11,11)\#}\Gamma_{11}$ and $A_{f_{44}}^{(11,11)\#}\Gamma_{11}$) are estimable.

For a design satisfying Condition (A) (or (B)), let

$$\tilde{P}_\beta^{*r}=\tilde{P}_\beta^r \quad \text{for } \beta=0,1 \quad (\text{or } \tilde{P}_0^{**r}=\tilde{P}_0^r),$$

$$\tilde{P}_f^{*t}=\tilde{P}_f^t \quad (\text{or } \tilde{P}_f^{**t}=\tilde{P}_f^t),$$

$$\tilde{P}_e^*=I_N-\left\{\sum_{r=0}^{5}\tilde{P}_0^{*r}+\sum_{s=0}^{2}\tilde{P}_1^{*s}+\sum_{t=0}^{5}\tilde{P}_f^{*t}\right\}$$

$$\left(\text{or } \tilde{P}_e^{**}=I_N-\left\{\sum_{r=0}^{5}\tilde{P}_0^{**r}+\sum_{t=0}^{5}\tilde{P}_f^{**t}\right\}\right),$$

where $v(m)-\varphi_2<N\leqslant v(m)$ (or $v(m)-(\varphi_2+3\varphi_1)<N\leqslant v(m)-\varphi_2$). Then for a design satisfying Condition (A) (or (B)).

$$\operatorname{rank}(\tilde{P}_\beta^{*r})=\varphi_\beta \quad \text{for } \beta=0,1 \quad (\text{or } \operatorname{rank}(\tilde{P}_0^{**r})=\varphi_0),$$

$$\operatorname{rank}(\tilde{P}_f^{*t})=\varphi_f \quad (\text{or } \operatorname{rank}(\tilde{P}_f^{**t})=\varphi_f),$$

$$\operatorname{rank}(\tilde{P}_e^*)=N-v(m)+\varphi_2 \quad (\text{or } \operatorname{rank}(\tilde{P}_e^{**})=N-v(m)+\varphi_2+3\varphi_1).$$

Theorem 4.6. *For a design satisfying Condition* (A) (*or* (B)),

$$\tilde{y}^*(T)'\tilde{y}^*(T)=\sum_{r=0}^{5}\tilde{S}_0^{*r}+\sum_{s=0}^{2}\tilde{S}_1^{*s}+\sum_{t=0}^{5}\tilde{S}_f^{*t}+\tilde{S}_e^*$$

$$\left(\text{or } \tilde{y}^{**}(T)'\tilde{y}^{**}(T)=\sum_{r=0}^{5}\tilde{S}_0^{**r}+\sum_{t=0}^{5}\tilde{S}_f^{**t}+\tilde{S}_e^{**}\right),$$

where

$$\tilde{S}_{\beta}^{*r}=\tilde{y}^*(T)'\tilde{P}_{\beta}^{*r}\tilde{y}^*(T) \quad \textit{for } \beta=0,1 \quad (\textit{or } \tilde{S}_0^{**r}=\tilde{y}^{**}(T)'\tilde{P}_0^{**r}\tilde{y}^{**}(T)),$$

$$\tilde{S}_f^{*t}=\tilde{y}^*(T)'\tilde{P}_f^{*t}\tilde{y}^*(T) \quad (\textit{or } \tilde{S}_f^{**t}=\tilde{y}^{**}(T)'\tilde{P}_f^{**t}\tilde{y}^{**}(T)),$$

$$\tilde{S}_e^{*}=\tilde{y}^*(T)'\tilde{P}_e^{*}\tilde{y}^*(T) \quad (\textit{or } S_e^{**}=\tilde{y}^{**}(T)'\tilde{P}_e^{**}\tilde{y}^{**}(T)).$$

Theorem 4.7. *An unbiased estimator of $\tilde{\sigma}^{*2}$ (or $\tilde{\sigma}^{**2}$) for a design satisfying Condition* (A) (*or* (B)) *is given by*

$$\hat{\tilde{\sigma}}^{*2}=\tilde{S}_e^{*}/\{N-v(m)+\varphi_2\} \quad (\textit{or } \hat{\tilde{\sigma}}^{**2}=\tilde{S}_e^{**}/\{N-v(m)+\varphi_2+3\varphi_1\}).$$

Using an argument similar to the one used in Section 4.3, the noncentrality parameters of the quadratic forms $\tilde{y}^*(T)'\tilde{P}_0^{*r}\tilde{y}^*(T)/\tilde{\sigma}^{*2}$ $(0\leqslant r\leqslant 5)$ (or $\tilde{y}^{**}(T)'\tilde{P}_0^{**r}\tilde{y}^{**}(T)/\tilde{\sigma}^{**2}$ $(0\leqslant r\leqslant 5)$), $\tilde{y}^*(T)'\tilde{P}_1^{*s}\tilde{y}^*(T)/\tilde{\sigma}^{*2}$ $(0\leqslant s\leqslant 2)$ and $\tilde{y}^*(T)'\tilde{P}_f^{*t}\tilde{y}^*(T)/\tilde{\sigma}^{*2}$ $(0\leqslant t\leqslant 5)$ (or $\tilde{y}^{**}(T)'\tilde{P}_f^{**t}\tilde{y}^{**}(T)/\tilde{\sigma}^{**2}$ $(0\leqslant t\leqslant 5)$) of a design satisfying Condition (A) (or (B)) are, respectively, given by

$$\tilde{\lambda}_0^{*r}/\tilde{\sigma}^{*2}=\sum_{a_1a_2}^{(r)}\sum_{b_1b_2}^{(r)}\{c_0^r(a_1a_2,b_1b_2)/\tilde{\sigma}^{*2}\}\Gamma'_{a_1a_2}A_0^{(a_1a_2,b_1b_2)\#}\Gamma_{b_1b_2} \quad \text{for } 0\leqslant r\leqslant 5$$

$$\left(\text{or } \tilde{\lambda}_0^{**r}/\tilde{\sigma}^{**2}=\sum_{a_1a_2}^{(r)}\sum_{b_1b_2}^{(r)}\{c_0^r(a_1a_2,b_1b_2)/\tilde{\sigma}^{**2}\}\Gamma'_{a_1a_2}A_0^{(a_1a_2,b_1b_2)\#}\Gamma_{b_1b_2} \quad \text{for } 0\leqslant r\leqslant 5\right),$$

$$\tilde{\lambda}_1^{*s}/\tilde{\sigma}^{*2}=\sum_{c_1c_2}^{*(s)}\sum_{d_1d_2}^{*(s)}\{c_1^s(c_1c_2,d_1d_2)/\tilde{\sigma}^{*2}\}\Gamma'_{c_1c_2}A_1^{(c_1c_2,d_1d_2)\#}\Gamma_{d_1d_2} \quad \text{for } 0\leqslant s\leqslant 2,$$

$$\tilde{\lambda}_f^{*t}/\tilde{\sigma}^{*2}=\sum_{u_1u_2;u}^{**(t)}\sum_{v_1v_2;v}^{**(t)}\{c_f^t((u_1u_2;u),(v_1v_2;v))/\tilde{\sigma}^{*2}\}\Gamma'_{u_1u_2}A_{fuv}^{(u_1u_2,v_1v_2)\#}\Gamma_{v_1v_2} \quad \text{for } 0\leqslant t\leqslant 5$$

$$\left(\text{or } \tilde{\lambda}_f^{**t}/\tilde{\sigma}^{**2}=\sum_{u_1u_2;u}^{**(t)}\sum_{v_1v_2;v}^{**(t)}\{c_f^t((u_1u_2;u),(v_1v_2;v))/\tilde{\sigma}^{**2}\}\times\Gamma'_{u_1u_2}A_{fuv}^{(u_1u_2,v_1v_2)\#}\Gamma_{v_1v_2} \quad \text{for } 0\leqslant t\leqslant 5\right),$$

where $c_\beta^r(a_1a_2,b_1b_2)$'s and $c_f^t((u_1u_2;u),(v_1v_2;v))$'s are given in Sect. 4.3. We summarize the results in Table 4 (or 5) for a design satisfying Condition (A) (or (B)).

For a design satisfying Condition (A) (or (B)), let $\tilde{H}_0^{*i\cdots 5}=\tilde{H}_0^{i\cdots 5}$ $(0\leqslant i\leqslant 5)$ and $\tilde{K}_0^{*5}=\tilde{K}_0^5$, (or $\tilde{H}_0^{**i\cdots 5}=\tilde{H}_0^{i\cdots 5}$ $(0\leqslant i\leqslant 5)$ and $\tilde{K}_0^{**5}=\tilde{K}_0^5$), $\tilde{H}_1^{*j\cdots 2}=\tilde{H}_1^{j\cdots 2}$ $(0\leqslant j\leqslant 2)$ and $\tilde{K}_1^{*2}=\tilde{K}_1^2$, and $\tilde{H}_f^{*k\cdots 5}=\tilde{H}_f^{k\cdots 5}$ $(0\leqslant k\leqslant 5)$ and $\tilde{K}_f^{*5}=\tilde{K}_f^5$ (or $\tilde{H}_f^{**k\cdots 5}=\tilde{H}_f^{k\cdots 5}$

Table 4
ANOVA of 3^m-BFF designs of resolution IV satisfying Condition (A)

Source	SS	d.f.	Noncentrality parameters
$A_0^{(00,00)\#}\Gamma_{00}$	$\tilde{S}_0^{*0}$	φ_0	$\tilde{\lambda}_0^{*0}/\tilde{\sigma}^{*2}$
$A_0^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_0^{*1}$	φ_0	$\tilde{\lambda}_0^{*1}/\tilde{\sigma}^{*2}$
$A_0^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_0^{*2}$	φ_0	$\tilde{\lambda}_0^{*2}/\tilde{\sigma}^{*2}$
$A_0^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_0^{*3}$	φ_0	$\tilde{\lambda}_0^{*3}/\tilde{\sigma}^{*2}$
$A_0^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_0^{*4}$	φ_0	$\tilde{\lambda}_0^{*4}/\tilde{\sigma}^{*2}$
$A_0^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_0^{*5}$	φ_0	$\tilde{\lambda}_0^{*5}/\tilde{\sigma}^{*2}$
$A_1^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_1^{*0}$	φ_1	$\tilde{\lambda}_1^{*0}/\tilde{\sigma}^{*2}$
$A_1^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_1^{*1}$	φ_1	$\tilde{\lambda}_1^{*1}/\tilde{\sigma}^{*2}$
$A_1^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_1^{*2}$	φ_1	$\tilde{\lambda}_1^{*2}/\tilde{\sigma}^{*2}$
$A_{f_{11}}^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_f^{*0}$	φ_f	$\tilde{\lambda}_f^{*0}/\tilde{\sigma}^{*2}$
$A_{f_{11}}^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_f^{*1}$	φ_f	$\tilde{\lambda}_f^{*1}/\tilde{\sigma}^{*2}$
$A_{f_{22}}^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_f^{*2}$	φ_f	$\tilde{\lambda}_f^{*2}/\tilde{\sigma}^{*2}$
$A_{f_{22}}^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_f^{*3}$	φ_f	$\tilde{\lambda}_f^{*3}/\tilde{\sigma}^{*2}$
$A_{f_{33}}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^{*4}$	φ_f	$\tilde{\lambda}_f^{*4}/\tilde{\sigma}^{*2}$
$A_{f_{44}}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^{*5}$	φ_f	$\tilde{\lambda}_f^{*5}/\tilde{\sigma}^{*2}$
Error	$\tilde{S}_e^{*}$	$N-v(m)+\binom{m-1}{2}$	
Total	$\tilde{y}(T)'\tilde{y}^{*}(T)$	N	

Table 5
ANOVA of 3^m-BFF designs of resolution IV satisfying Condition (B)

Source	SS	d.f.	Noncentrality parameters
$A_0^{(00,00)\#}\Gamma_{00}$	$\tilde{S}_0^{**0}$	φ_0	$\tilde{\lambda}_0^{**0}/\tilde{\sigma}^{**2}$
$A_0^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_0^{**1}$	φ_0	$\tilde{\lambda}_0^{**1}/\tilde{\sigma}^{**2}$
$A_0^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_0^{**2}$	φ_0	$\tilde{\lambda}_0^{**2}/\tilde{\sigma}^{**2}$
$A_0^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_0^{**3}$	φ_0	$\tilde{\lambda}_0^{**3}/\tilde{\sigma}^{**2}$
$A_0^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_0^{**4}$	φ_0	$\tilde{\lambda}_0^{**4}/\tilde{\sigma}^{**2}$
$A_0^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_0^{**5}$	φ_0	$\tilde{\lambda}_0^{**5}/\tilde{\sigma}^{**2}$
$A_{f_{11}}^{(10,10)\#}\Gamma_{10}$	$\tilde{S}_f^{**0}$	φ_f	$\tilde{\lambda}_f^{**0}/\tilde{\sigma}^{**2}$
$A_{f_{11}}^{(01,01)\#}\Gamma_{01}$	$\tilde{S}_f^{**1}$	φ_f	$\tilde{\lambda}_f^{**1}/\tilde{\sigma}^{**2}$
$A_{f_{22}}^{(20,20)\#}\Gamma_{20}$	$\tilde{S}_f^{**2}$	φ_f	$\tilde{\lambda}_f^{**2}/\tilde{\sigma}^{**2}$
$A_{f_{22}}^{(02,02)\#}\Gamma_{02}$	$\tilde{S}_f^{**3}$	φ_f	$\tilde{\lambda}_f^{**3}/\tilde{\sigma}^{**2}$
$A_{f_{33}}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^{**4}$	φ_f	$\tilde{\lambda}_f^{**4}/\tilde{\sigma}^{**2}$
$A_{f_{44}}^{(11,11)\#}\Gamma_{11}$	$\tilde{S}_f^{**5}$	φ_f	$\tilde{\lambda}_f^{**5}/\tilde{\sigma}^{**2}$
Error	$\tilde{S}_e^{**}$	$N-v(m)+2m^2-6m+1$	
Total	$\tilde{y}^{**}(T)'\tilde{y}^{**}(T)$	N	

$(0 \leqslant k \leqslant 5)$ and $\tilde{K}_f^{**5} = \tilde{K}_f^5$). Then all test satistics for the nested method

$$\frac{\tilde{S}_0^{*5}/\varphi_0}{\tilde{S}_e^*/\{N - v(m) + \varphi_2\}} \quad (= \tilde{F}_0^{*5}, \text{ say}),$$

$$\frac{\tilde{S}_0^{*4}/\varphi_0}{(\tilde{S}_e^* + \tilde{S}_0^{*5})/\{N - v(m) + \varphi_2 + \varphi_0\}} \quad (= \tilde{F}_0^{*45}, \text{ say}), \ldots, \text{ and}$$

$$\frac{\tilde{S}_0^{*i}/\varphi_0}{(\tilde{S}_e^* + \tilde{S}_0^{*i+1} + \cdots + \tilde{S}_0^{*5})/\{N - v(m) + \varphi_2 + (5-i)\varphi_0\}} \quad (= \tilde{F}_0^{*i \cdots 5}, \text{ say})$$

$$\Bigg(\text{or } \frac{\tilde{S}_0^{**5}/\varphi_0}{\tilde{S}_e^{**}/\{N - v(m) + \varphi_2 + 3\varphi_1\}} \quad (= \tilde{F}_0^{**5}, \text{ say}),$$

$$\frac{\tilde{S}_0^{**4}/\varphi_0}{(\tilde{S}_e^{**} + \tilde{S}_0^{**5})/\{N - v(m) + \varphi_2 + 3\varphi_1 + \varphi_0\}} \quad (= \tilde{F}_0^{**45}, \text{ say}), \ldots, \text{ and}$$

$$\frac{\tilde{S}_0^{**i}/\varphi_0}{(\tilde{S}_e^{**} + \tilde{S}_0^{**i+1} + \cdots + \tilde{S}_0^{**5})/\{N - v(m) + \varphi_2 + 3\varphi_1 + (5-i)\varphi_0\}} \quad (= \tilde{F}_0^{**\, i \cdots 5}, \text{ say})\Bigg),$$

$$\frac{\tilde{S}_1^{*2}/\varphi_1}{\tilde{S}_e^*/\{N - v(m) + \varphi_2\}} \quad (= \tilde{F}_1^{*2}, \text{ say}),$$

$$\frac{\tilde{S}_1^{*1}/\varphi_1}{(\tilde{S}_e^* + \tilde{S}_1^{*2})/\{N - v(m) + \varphi_2 + \varphi_1\}} \quad (= \tilde{F}_1^{*12}, \text{ say}), \ldots, \text{ and}$$

$$\frac{\tilde{S}_1^{*j}/\varphi_1}{\tilde{S}_e^* + \tilde{S}_1^{*j+1} + \cdots + \tilde{S}_1^{*2})/\{N - v(m) + \varphi_2 + (2-j)\varphi_1\}} \quad (= \tilde{F}_1^{*j \cdots 2}, \text{ say})$$

$$\frac{\tilde{S}_f^{*5}/\varphi_f}{\tilde{S}_e^*/\{N - v(m) + \varphi_2\}} \quad (= \tilde{F}_f^{*5}, \text{ say}),$$

$$\frac{\tilde{S}_f^{*4}/\varphi_f}{(\tilde{S}_e^* + \tilde{S}_f^{*5})/\{N - v(m) + \varphi_2 + \varphi_f\}} \quad (= \tilde{F}_f^{*45}, \text{ say}), \cdots, \text{ and}$$

$$\frac{\tilde{S}_f^{*k}/\varphi_f}{(\tilde{S}_e^* + \tilde{S}_f^{*k+1} + \cdots + \tilde{S}_f^{*5})/\{N - v(m) + \varphi_2 + (5-k)\varphi_f\}} \quad (= \tilde{F}_f^{*k \cdots 5}, \text{ asy})$$

$$\Bigg(\text{or } \frac{\tilde{S}_f^{**5}/\varphi_f}{\tilde{S}_e^{**}/\{N - v(m) + \varphi_2 + 3\varphi_1\}} \quad (= \tilde{F}_f^{**5}, \text{ say}),$$

$$\frac{\tilde{S}_f^{**4}/\varphi_f}{(\tilde{S}_e^{**} + \tilde{S}_f^{**5})/\{N - v(m) + \varphi_2 + 3\varphi_1 + \varphi_f\}} \quad (= \tilde{F}_f^{**45}, \text{ say}), \ldots, \text{ and}$$

$$\frac{\tilde{S}_f^{**k}/\varphi_f}{(\tilde{S}_e^{**} + \tilde{S}_f^{**k+1} + \cdots + \tilde{S}_f^{**5})/\{N - v(m) + \varphi_2 + 3\varphi_1 + (5-k)\varphi_f\}} \quad (= \tilde{F}_f^{**k \cdots 5}, \text{ say})\Bigg),$$

have F distributions and this procedure is continued until a significant test is obtained for each case. We note that $\tilde{F}_0^{*i\cdots 5}$ $(0 \leqslant i \leqslant 5)$ (or $\tilde{F}_0^{**i\cdots 5}(0 \leqslant i \leqslant 5)$), $\tilde{F}_1^{*j\cdots 2}$ $(0 \leqslant j \leqslant 2)$ and $\tilde{F}_f^{*k\cdots 5}$ $(0 \leqslant k \leqslant 5)$ (or $\tilde{F}_f^{**k\cdots 5}$ $(0 \leqslant k \leqslant 5)$) are noncentral F distributions with φ_0 and $\{N - v(m) + \varphi_2 + (5-i)\varphi_0\}$ d.f. (or φ_0 and $\{N - v(m) + \varphi_2 + 3\varphi_1 + (5-i)\varphi_0\}$ d.f.), φ_1 and $\{N - v(m) + \varphi_2 + (2-j)\varphi_1\}$ d.f., and φ_f and $\{N - v(m) + \varphi_2 + (5-i)\varphi_k\}$ d.f. (or φ_f and $\{N - v(m) + \varphi_2 + 3\varphi_1 + (5-k)\varphi_f\}$ d.f.) and noncentrality parameters $\tilde{\lambda}_0^{*i}/\tilde{\sigma}^{*2}$ (or $\tilde{\lambda}_0^{**i}/\tilde{\sigma}^{**2}$), $\tilde{\lambda}_1^{*j}/\tilde{\sigma}^{*2}$ and $\tilde{\lambda}_f^{*k}/\tilde{\sigma}^{*2}$ (or $\tilde{\lambda}_f^{**k}/\tilde{\sigma}^{**2}$) if $\tilde{H}_0^{*i\cdots 5}$ (or $\tilde{H}_0^{**c\cdots 5}$), $\tilde{H}_1^{*j\cdots 2}$ and $\tilde{H}_f^{*k\cdots 5}$ (or $\tilde{H}_f^{**k\cdots 5}$) are false, respectively.

5. Related unsolved problems

A relationship between a BA$(N, m, s, 2l; \{\mu_{i_0 i_1 \cdots i_{s-1}}\})$ and an s^m-BFF design of resolution $2l+1$ was presented by Kuwada and Nishii [20]. Using an algebraic structure similar to the one used in the case of a 3^m-BFF design of resolution V, Kuwada and Nishii [21] obtained the characteristic polynomial of the information matrix of an s^m-BFF design of resolution $V_{p,q}$, in particular, when $p = q = s - 1$, it is of resolution V. Thus the author believes that the ANOVA and the testing of hypotheses of an s^m-BFF design of resolution V can be obtained by using a method similar to that used for a 3^m-BFF design of resolution V, as shown in Section 4.3.

Problem 5.1. Using the algebraic structure, construct the ANOVA table of an s^m-BFF design of resolution V derived from a BA$(N, m, s, 4; \{\mu_{i_0 i_1 \cdots i_{s-1}}\})$ and obtain the test statistics for the nested procedure as in Section 4.3.

Let $H_\beta^i(\beta \leqslant i \leqslant l,\ 0 \leqslant \beta \leqslant l)$ be the hypotheses given in Section 3.3. Then one method for testing $H_\beta^{i\cdots l}$ against K_β^l would be to accept the hypothesis if each of the $(l - i + 1)$ hypotheses H_β^k $(i \leqslant k \leqslant l)$ against K_β^l is accepted separately. In an experimental design, the orthogonality of a design relative to a general linear model and linear hypotheses $H_\beta^l, \ldots, H_\beta^i$ was studied by several authors (e.g., Darroch and Silvey [4]). For a 3^m-BFF design of resolution V, we can consider the testing hypotheses as above.

Problem 5.2. Find a necessary and sufficient condition for a 2^m-BFF design of resolution $2l+1$ to be orthogonal relative to a general model and linear hypotheses $H_\beta^l, \ldots, H_\beta^i$ $(\beta \leqslant i \leqslant l,\ 0 \leqslant \beta \leqslant l)$.

Problem 5.3. Find a necessary and sufficient condition for a 3^m-BFF design of resolution V to be orthogonal relative to a general model and linear hypotheses $\tilde{H}_0^5, \ldots, \tilde{H}_0^i$ $(\tilde{H}_1^2, \ldots, \tilde{H}_1^j$ or $\tilde{H}_f^5, \ldots, \tilde{H}_f^k)$.

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